Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas conſtituunt, ortu & demonſtrationibus.

Secundus ARCHITECTONICVS, ſeu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel ſolido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Eſſentiâ earumque generibus in Mundo; præſertim de Harmonia radiorum, ex corporibus cœleſtibus in Terram deſcendentibus, eiuſque effectu in Natura ſeu Anima ſublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis abſolutiſſimis motuum cœleſtium, ortuque Eccentricitatum ex proportionibus Harmonicis.

Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxonienſis ſpeculationibus Harmonicis, operi de Macrocoſmo & Microcoſmo inſertis.

Cum S. C. M^tis. Priuilegio ad annos XV.

Lincii Auſtriæ,

Sumptibus GODOFREDI TAMPACHII Bibl. Francof.

Excudebat IOANNES PLANCVS.

ANNO M. DC. XIX.

THE FIVE BOOKS OF
Johannes Kepler's
HARMONY OF THE WORLD

of which

The first is GEOMETRICAL, on the origin and constructions of the regular figures which establish the harmonic proportions;

The second is ARCHITECTONIC, or comes from the GEOMETRY OF FIGURES, on the congruence of the regular figures in the plane or solid;

The third is specifically HARMONIC, on the origin of the harmonic proportions in the figures, and on the nature and distinguishing features of matters relating to music, contrary to the ancients;

The fourth is METAPHYSICAL, PSYCHOLOGICAL, AND ASTROLOGICAL, on the mental essence of the harmonies and on the types of them in the world, especially on the harmony of the rays which descend from the heavenly bodies to the Earth, and on its effect on Nature or the sublunary and human soul;

The fifth is ASTRONOMICAL AND METAPHYSICAL, on the most perfect harmonies of the celestial motions, and the origin of the eccentricities in the harmonic proportions.

The Appendix contains a comparison of this work with Book III of the *Harmony* of Claudius Ptolemy and with the harmonic speculations of Robert of the Floods, called Fludd, the Oxford physician, inserted in his work on the macrocosm and microcosm.

With Imperial Privilege for fifteen years.

Printed at the expense of GOTTFRIED TAMPACH,
bookseller of Frankfurt, by
JOHANNES PLANCK,
AT LINZ
In the year 1619.

TO THE MOST SERENE AND POWERFUL PRINCE AND LORD JAMES, KING OF GREAT BRITAIN, FRANCE, AND IRELAND, DEFENDER OF THE FAITH, ETC.,

MY MOST MERCIFUL LORD:

The reasons for my transferring these books on Harmony, which are to be published to the world, away from the court of the most august Emperor, my lord, from his kingdoms and hereditary Austrian provinces, and in fact away from Germany, across the sea and bringing them into your most serene presence, glorious King, were partly in the present and partly old.

For, first, I did not think it inconsistent with my duty that as one who receives a salary from Caesar[1] for mathematics I should therefore show to the outside world also what farsighted provision the Prince of this Christian state made for such divine studies, and that he should understand from the uninterrupted progress of the ornaments of peace throughout these provinces that the rumor of civil war would undoubtedly soon be extinguished together with its reality,[2] and that this slightly too harsh discord, as in an emotional melody, is on the very point of resolution into a pleasing cadence. Who indeed would be a more worthy assessor of the imperial benevolence than a great king? What more appropriate patron could I choose for a work on the harmony of the heavens, with its savor of Pythagoras and Plato, than that King who has borne witness to his study of Platonic learning by domestic tokens,[3] which we know also from the public veneration

[1] Caesar, originally the family name of the earliest Roman emperors, became a title of the later rulers of the original Roman Empire. The Holy Roman Emperor, though he was only nominally the successor of the ancient Roman emperors, was in Latin referred to by the title of Caesar to emphasize his claim to carry on the Roman tradition. The emperor in whose service Kepler had succeeded Tycho Brahe was Rudolph II, whose chief seat was in Prague. He had been obliged to abdicate in favor of the Archduke Matthias in 1611, and died in 1612. Matthias continued to show favor to Kepler, and is probably the Caesar meant here. However, Kepler had meanwhile felt it necessary to leave Prague and move to a new post in Linz after the death of Rudolph in 1612. Kepler was invited to move to England in 1618 by Sir Henry Wotton, English ambassador to the emperor's court, but he preferred to remain in Central Europe. Matthias died in the course of 1619, and was succeeded by the Archduke Ferdinand as Ferdinand II, no doubt after the *Harmonice Mundi* had gone to press.

[2] There were many wars in the German-speaking world during Kepler's lifetime, but here he is no doubt referring to the civil war which had brought about Rudolph's abdication. In 1611 the emperor's cousin, Leopold, bishop of Passau, brought up an army, supposedly in support of Rudolph, occupied part of Prague, and had to be bribed to withdraw.

[3] James Stewart (1566-1625) became king of Scotland at the age of thirteen months when his Catholic mother, Mary, Queen of Scots, was forced to abdicate in 1567. James was brought up by his guardian as a Protestant, though unlike Kepler

of his subjects? Who when still a youth deemed the astronomy of Tycho Brahe,[4] on which this work depends, worthy of the ornaments of his talent? Who, indeed, on becoming a man, when he was at the helm of his kingdom, marked the excesses of astrology with public censure?[5] —which are in fact very clearly revealed in Book IV of this work, where the true bases of the effects of the stars are disclosed. Thus nobody can have any doubt that you will have complete understanding of the whole of this work and of all its parts.

Yet my more important reason for this dedication from of old is the following. I first conceived in my mind the material of the work a little less than twenty years ago,[6] and gave it its title, when I had not yet discovered the proper motions of the planets, though nature's instinct declared that the harmonies were in them. Even then I intended to place the work, if ever it were to prosper and be completed, under your Majesty's patronage; and I bore witness of this, as it were, vow of mine time and again to your representatives at the imperial court. The reasons for thinking of this patronage for my Harmony were supplied by that manifold dissonance in human affairs, which is indeed obvious, so that it cannot fail to offend, though it is compounded of melodic and distinct intervals, the nature of which is to mollify the hearing in the midst of the dissonance with the promise of the pleasing consonance which is to succeed, and to sustain it in the expectation of the same. For indeed it was a belief worthy of a Christian man that it was God who regulated all the melody of human life, and a patience worthy of the greatness of God not to be offended by the diffuseness of the dissonances, nor to abandon hopes, reflecting that it is not the providence of God which acts slowly, but the space of our individual lifetimes which is so swiftly fleeting. For my part I learned from the sacred oracles that all things have been destined by God for

as a Calvinist. His mother was executed at the order of Elizabeth Tudor, Queen of England, in 1587. James also became King of England on the death of Elizabeth in 1603. His intellectual interests —literary, philosophical, and theological —were indeed deeper and more extensive than those of most kings, though not particularly Platonic. As well as a number of books on various parts of the Bible, on monarchy, and on poetry, he published attacks on witchcraft and on the smoking of tobacco. Another link between James and Central Europe was the marriage of his daughter Elizabeth Stuart in 1613 to the Palatine Frederick, the leader of the German Protestants. In 1619 (the year of publication of the *Harmonice Mundi)* Frederick accepted the crown of Bohemia from the Protestants of that country, who were in revolt against the Catholic emperor and his government. However, he and his wife (who was consequently known as the Winter Queen) were driven from Bohemia after his defeat by the emperor's army in 1620. Although Kepler wrote his dedication before those events, he may have been hoping for further patronage from Frederick.

[4] King James visited Denmark, another Protestant country, in 1589-1590 after his marriage to Anne of Denmark. During the visit he presented a cup to the University of Copenhagen and went to see Tycho Brahe's observatory.

[5] James attacked astrology in his own writing.

[6] The intention was announced to Herwart von Hohenburg in a letter of 14 December 1599 (KGW 14, p. 100). See our Introduction, p. xv and note 17.

definite and salutary purposes, even those dissonances, for elucidating and recommending the pleasing nature of consonance. However, there was a reason why my longings prompted me to look for some basis for reconstructing consonance from your Davidic harp, glorious King, though it is not appropriate to explain it more extensively here, lest I should seem to spurn the advice of the prudent. Yet no-one should prevent me from touching on this feature, long recognized by the whole world, of the glory of your deeds, that having obtained the kingdom of England by inheritance and by the agreement of the people you soon gave it the name, in common with the Kingdom of Scotland, of Great Britain;[7] from the combination of both provinces you produced one kingdom and one harmony (for what else is a kingdom but a harmony?); you removed in the happiest way the hereditary discord between two extremely hostile nations; and you completely removed the memory of the frequent and most bloody massacres with which, as if by black marks, the passage of the ages had been blotted. This, your work at home, seemed to me to contain a not untrustworthy omen (among other more weighty matters) that abroad also, as a King among kings, as Defender of the Faith among faithful followers of Christ,[8] you would perform some greater and more excellent, and also more lasting work. Indeed, I followed that up both by my silent vows and by a public prognostication, in the book on the new star,[9] which burnt like a fiery coal (a verse well known in Scotland). So, as if what I wished for and foretold to so praiseworthy an adjudicator were now completed, I set myself all the more resolutely to chant the cosmic harmonies at some future time.

I should wish here for the three part public dissonance of clamoring voices to show me a little more moderation so that I could make the results of my own thinking publicly heard, in which case would the outcome seem to fulfil my vows? What wounds to my person treated by what harmonies, by what physician? And just as I painted it in lively colors in my book on the new star, long ago? But what use will it be, if in striving for harmony with my private clamor I do not overcome the public roaring, through the weakness of my support, and in addition I increase the annoyance of the absurd chorus to my ears? For my part I must confess — ah ! what sorrow— that the criss-cross wound is still swollen, or if we prefer a more sacred and more felicitous

[7] See note 3 above. Although James was both king of Scotland and king of England, and used the term "Great Britain," the two kingdoms remained at least nominally separate from each other until the Act of Union of 1707.

[8] The title "Defender of the Faith," meaning defender of the Roman Catholic faith, had been awarded by the Pope to King Henry VIII of England in recognition of a pamphlet which he had published in defense of that faith before his breach with Rome. The title, however, has continued to be used by monarchs of England, and later of the United Kingdom, even by those who like James VI and I were Protestants, up to the present day.

De Stella Nova, Prague, 1606 (KGW 1).

word, the cross-shaped wound,[10] is swollen, I say, with its multiple lip; and though none of them winks at it, the medicine has so far been useless, and jeered at from all sides, because the physician, to force a deceptive medicine on a crazy patient, makes many pretenses, and many embellishments, which seem to stray far from sensible reasoning. Yet I am invigorated by the very thought that the supreme Healer of our wounds is sure in his art, and applies no remedy in vain. Therefore, he who embarked on that care, who has now brought forward, has now shown to the world that confirmation, but meanwhile encounters impairment through public calamities, inasmuch as since the flesh which is rotting and unfit, has been devoured, I mean that of charity which is defunct, some sense of regret has descended to the depth of the living flesh; the same will undoubtedly soon use means of alleviation to reduce the swelling, so that indeed there may be room for that confirmation, and at last this enduring dissonance (to revert to the metaphor which was suggested) will end in pure and abiding harmony. I am strengthened in this hope even against hope not only by the success of my speculations on harmony, inasmuch as my good fortune prevails over my audacity in searching for so long; but also by this fact in particular, that among the original long-standing requirements for the completion of the work, I have seen your Royal Majesty also, for whom I had intended the patronage of this work before it was begun, safe and flourishing; and I shall not cease to entreat God the Author of peace and concord with my devout prayers to watch over the safety of your life and your Royal Majesty until that longed-for result.

In the meantime I supplicate your Reverend Majesty to look upon this work on harmony, dedicated to your name, with a kindly eye; to consider honest and good this expression of my most devoted affection towards yourself; to delight your royal mind indeed by the contemplation of the works of God, insofar as the necessary business of kingship allows; to strengthen and stir up in yourself by the examples of the brilliance of concord in the visible works of God the zeal for concord and for peace in church and state; and finally to deem me and my studies worthy of your most clement royal patronage.

Written at Linz on the Danube on the thirteenth of February in the year of the western era 1619.

In reverence to your most serene Royal Majesty with all submission, the mathematician in Upper Austria of Matthias, Emperor and Archduke of his loyal orders,

JOHANNES KEPLER

[10] Kepler, though a Lutheran, was unable to accept the doctrine of the sacrament laid down in the Formula of Concord, and had therefore been excluded from the Lutheran communion. He had not been readmitted, in spite of an appeal to the Stuttgart Consistory which governed the Lutheran church at Linz, and later to the faculty of his old university at Tubingen through his former teacher Matthias Hafenreffer, now chancellor.

BOOK I

ON THE HARMONY OF THE WORLD

by

JOHANNES KEPLER

ON THE ORIGIN, CLASSES, ORDER, AND DISTINGUISHING FEATURES OF THE REGULAR FIGURES WHICH GIVE RISE TO HARMONIC PROPORTIONS, FOR THE SAKE OF KNOWLEDGE AND CONSTRUCTION OF THEM.

Proclus Diadochus in Book I of his Commentary on Euclid Book I.

"(Mathematics) contributes things of the greatest importance to the study of nature, both revealing the orderly nature of the reasoning, in accordance with which the WHOLE has been constructed, and so on, and showing that the simple and primary elements, by means of which the whole of the heaven was completed, having taken on the appropriate forms among its parts, are connected together with symmetry and regularity."

With Imperial privilege for fifteen years.

Printed by JOHANNES PLANCK at LINZ, in the year 1619.

Introduction

We must seek the causes of the harmonic proportions in the divisions of a circle into equal aliquot parts, which are made geometrically and knowably, that is, from the constructible regular plane figures. I thus considered that to start with it should be intimated that the features which distinguish geometrical objects to the mind are today, as far as is apparent from published books, totally unknown. In fact not even among the ancients is anyone found who has intimated that he knew exactly these specific distinguishing features of geometrical objects, except for Euclid and his commentator Proclus. Indeed in Pappus the Alexandrine and the ancients who follow him the division of problems into plane, solid, and linear is sufficiently appropriate for explaining the mental attitudes which arise in connection with every single part of the subject of geometry.[1] However it is both brief in words and applied to practice: no mention is made of theory, and yet unless we engage with our whole minds in the theory of this matter we shall never be able to take in the harmonic ratios. Proclus Diadochus in the four books which he published on the first book of Euclid explicitly played the part of a theoretical philosopher dealing with a mathematical subject. If he had left to us his commentaries on the tenth book of Euclid as well, he would both have freed our geometers from ignorance, if he had not been neglected, and relieved me totally from this toil of explaining the distinguishing features of geometrical objects. For from the very outset it is readily apparent that those distinctions between entities of the mind would have been known, since he established the basic principles of the whole essence of mathematics as the same which also pervade all entities and generate them all from themselves, that is to say the end and the endless, or the limit and the unlimited, recognizing the limit or boundary as the form, the unlimited as the matter of geometrical objects.

PROCLUS on the intellectual essence of geometrical objects.

For shape and proportion are properties of quantities, shape of individual quantities and proportion of quantities in combination. Shape is demarcated by limits, for it is by points that a straight line, by lines that a plane surface, by surfaces that a solid is bounded, circumscribed, and shaped. Therefore finite things which are circumscribed and shaped can also be grasped by the mind: infinite and unbounded things, insofar as they are such, can be held in by no bonds of knowledge, which is obtained from definitions, by no bonds of constructions. For shapes are in the archetype prior to their being in the product, in the divine mind prior to being in creatures, differently

[1] Kepler discusses this system of classifying problems in section XLVI below (see note 251).

indeed in respect of their subject, but the same in the form of their essence.-Therefore in quantities shape is a kind of mental essence of them, or understanding is their essential distinguishing feature. That is much clearer from the case of proportions. For since shape is demarcated by several limits, it comes about that on account of their being plural shape partakes of proportions. However what proportion is without the action of the mind is something which cannot be understood in any way. Hence by the same reasoning, one who gives limits to quantities as their essential basis supposes that quantities which have shapes have an intellectual essence. But there is no need for arguments: Proclus' whole book should be read. It will be sufficiently evident that the intellectual distinguishing features of geometrical objects were properly known to him, although he does not make such an open and conspicuous declaration of that point as a separate thing on its own, so as to put even a dozing reader in mind of it. For his eloquence flows as if in full flood, completely swathed in most plentiful doctrines of rather abstruse Platonic philosophy; and among them is this point, which is the single argument of this Book.

Petrus Ramus' unjust and ignorant criticism of Euclid.

However this our age has had no room hitherto for penetration to such hidden mysteries. The book of Proclus was read by Petrus Ramus,[2] but as far as the core of philosophy is concerned, it was despised and rejected equally with the tenth Book of Euclid; and he who had written a commentary on Euclid was repudiated and instructed to lose his voice, as if he had written a defense for him. Indeed the peevish anger of a hostile critic turns against Euclid as if he were on trial: the tenth Book of Euclid was condemned to the atrocious sentence of not being read, though if it were read and understood it could lay bare the secrets of philosophy. Read, I ask you, the words of Ramus, than which he has never uttered anything more unworthy of Ramus. *(Scholae Mathematicae (Mathematical Schools),* Book 21.) *"The material,"* he says, *"which is the subject of Book X, is conveyed in such a way that I have never found the same obscurity in literature or the arts — obscurity, I mean, not for the understanding of what Euclid is saying (for that may be quite clear to the unlearned and unlettered if they pay attention to it, that is only what is there and what is present in the text) but for thoroughly understanding and investigating what end and what purpose are intended for the work, what the kinds, species and distinguishing features of the objects under discussion are; for I have never read or heard anything so confused or involved. Furthermore the superstition of the Pythagoreans seems to have invaded this, so to speak, cave."*

But, my goodness, Ramus, if you had not believed that this book

[2] Petrus Ramus (Pierre de la Ramee, 1515-1572). Although he castigates Ramus, both here and in section XXV below, for having attacked Euclid, Kepler approved of some other elements in Ramus' philosophy, for instance his call for "an astronomy without hypotheses"— though Kepler's interpretation of Ramus' phrase almost certainly misses its intended meaning *(Astronomici nova,* Heidelberg, 1609, folio 1 *verso,* KGW 3, p. 6).

was too hard to understand, you would never have slandered it with the accusation of such obscurity. There is need for harder work, need for tranquillity, need for concentration, and above all for mental exertion, until you grasp the writer's intention. When the superior mind has struggled to that point, then at last, seeing that it has reached the light of truth, it is exultantly flooded with incredible pleasure, and in that, as it were, watch tower, it perceives with great precision the whole world and all the distinguishing features of its parts. But to you, who here act as the patron of ignorance, and to the common herd of men who snatch at profit from everything, divine or human, to you, I say, belong the phrases "prodigious sophisms," to you "Euclid incontinently abusing his leisure," to you "these subtleties have no place in geometry." Let your part be to carp at what you do not understand: for me, a hunter for the causes of things, no other paths to them had opened but in the tenth Book of Euclid.

Following Ramus, Lazarus Schoner in his *Geometry** confessed that he could see absolutely no use for the five regular solids in the world, until he perused my little book which I entitled *The Secret of the Universe,* in which I prove that the number and distances of the planets were taken from the five regular solids. See what damage Ramus the master did to Schoner the disciple. First Ramus, having read Aristotle thoroughly, who had refuted the Pythagorean philosophy on the properties of the elements as deduced from the five solids, at once conceived in his mind a contempt for the whole of the Pythagorean philosophy; and then, as he knew that Proclus was a member of the Pythagorean sect, he did not believe him when he asserted, which was quite true, that the ultimate aim of Euclid's work, to which absolutely all the propositions of all its books were related, was the five regular solids. Hence there arose in Ramus a very confident conviction that the five solids must be removed from the aim of the books of the *Elements* of Euclid.

With the aim of the work removed, as if the form were removed from a building, there was left a formless heap of propositions in Euclid, which Ramus attacked as if it were a fiend in all the twenty-eight books of his Schools, with great harshness of language, with great temerity, quite undeserved by so great a man. Schoner, following Ramus' convictions, himself also believed (of course) that the regular solids had no application; and not only that, but he also neglected or despised Proclus, following the judgement of Ramus. Yet from Proclus he could learn the application of the five solids both in the *Elements* of Euclid and in the structure of the world. In fact the disciple was more for-

The opinion of Lazarus Schonei on the five solid shapes.

[3] Kepler appears to be mistaken about this. There is no reference to him or his work in any of the works by Schoner we have been able to find. (Ramus himself is, indeed, dismissive of applications of the Platonic solids in physical theories.)

The only personal connection between Kepler and Schoner to emerge from Kepler's correspondence is a reference to Schoner's having written some letters on optics which are to be published in a book with one of Kepler's letters on the same subject (Kepler to Herwart von Hohenburg, 13 Jan 1606, KGW 15, p. 299, letter 368, line 161ff.).

tunate than the master, because he gratefully received my revelation of the application of the solids in the structure of the world, which Ramus had repudiated though it was taught by Proclus. For what if the Pythagoreans attributed these shapes to the elements, but not as I do to the spheres of the world? Ramus would have striven to undo this error over the true subject of the figures, as I have done; he would not have demolished this whole philosophy with one tyrannical word. What if the Pythagoreans put forward the same teaching as I do, and hid their doctrine by wrapping it up in words? Is not the Copernican form of the world found in Aristotle himself, falsely refuted by him under other names, as they called the Sun, Fire, and the Moon the Counter Earth? For suppose the disposition of the circles was the same according to the Pythagoreans as according to Copernicus, that the five solids were known, and the necessity for their fivefold number; and suppose that they all consistently taught that the five solids were the archetypes of the parts of the world. What a short step further it is for us to believe that their doctrine in the form of a riddle was read by Aristotle as if it had been refuted in the true sense of the words, when Aristotle read it as the Earth to which they allocated the cube, although they as it happened meant Saturn, the orbit of which was separated from Jupiter by the interposition of the cube. And the common herd ascribe rest to the Earth, whereas Saturn has been allocated a very slow motion which is very close to rest, so that among the Hebrews it got its name from the word "rest." Similarly Aristotle read it as the air to which the octahedron was given, whereas they as it happened meant Mercury, the orbit of which was enclosed by the octahedron; and Mercury is no less swift than the nimble air is held to be. As it happened, Mars was the interpretation given to the word "fire," which also had elsewhere the name Pyrois (fiery one) from fire; and to it the tetrahedron was given, perhaps because its orbit is enclosed by that figure. And under the disguise of water, to which the icosahedron is attributed, the star of Venus (as the one of which the course is contained within the icosahedron) could be hidden, because liquids are subject to Venus, and she herself is said to have risen from the sea foam, whence the name "Aphrodite." Lastly, the word "world" could signify the Earth, and that the dodecahedron is ascribed to the world, because the Earth's course is contained within that figure, and marked off into twelve sections of its length, as that figure is contained within twelve faces round its whole compass. Therefore that in the secrets of the Pythagoreans on this basis the five figures were distributed not among the elements, as Aristotle believed, but among the planets themselves is very strongly confirmed by the fact that Proclus tells us that the aim of geometry is to tell how the heaven has received appropriate figures for definite parts of itself.

The Pythagoreans' mystical interpretation of the five figures.

Nor is this yet the end of the damage which Ramus has inflicted on us. Consider the most ingenious of today's geometers, Snel, clearly a supporter of Ramus, in his preface to the *Problems* of Ludolph van

Willebrord Snel's opinion on binomials.

Ceulen.[4] First he says, "That division of the inexpressibles into thirteen kinds is useless for application." I concede that, if he is to recognize no application unless it is in everyday life, and if there is to be no application of the study of nature to life. But why does he not follow Proclus, whom he mentions, and who recognizes that there is some greater good in geometry than those of the arts which are necessary for living? In that case in fact the application of the tenth Book in deciding the kinds of figures would have been evident. Snel mentions geometrical authors who are said to make no use of the tenth Book of Euclid.[5] Of course all of them deal with either linear or solid problems, and in connection with such figures or quantities as have no purpose within themselves, but obviously aim at other applications, and would not be investigated otherwise. But the regular figures are investigated on their own account as archetypes, have their own perfection within themselves, and are among the subjects of plane problems, notwithstanding the fact that a solid is also enclosed by plane faces. In the same way the material of the tenth Book also relates chiefly to plane surfaces. Why then should those of varying kinds be mentioned? Or why should the goods which Codrus did not buy to feed his belly with them, but which Cleopatra bought to ornament her ears, be reckoned cheap? "Is it only a cross fastened to our talents?"[6] I say, to those who molest the inexpressibles with numbers, that is by expressing them. But I deal with those kinds not with numbers, not by algebra, but by mental processes of reasoning, because of course I do not need them in order to draw up accounts of merchandise, but to explain the causes of things. He considers that such subtleties should be kept out of a "primer," and hidden away in a library. He plays completely the part of the faithful disciple of Ramus, and shows no mean judgement in placing his effort. Ramus removed the form from Euclid's edifice, and tore down the coping stone, the five solids. By their removal every joint was loosened, the walls stand split, the arches threatening to collapse. Snel therefore takes away the stonework as well, seeing that there is no application for it except for the stability of the house which was joined together under the five solids. How fortunate is the disciple's understanding, and how dexterously did he learn from

[4] The comment Kepler quotes comes from the Introduction to *Ludophi a Ceulen, Variorum problematum libri IV, a Willibrordo Snellio e vernaculo in latinum translati, ac varijs locis demonstrationibus audi et illustrati,* Leiden, 1615.

[5] Willibrord Snel (1581-1626), who was personally acquainted with both Kepler and Tycho, was the son of Rudolf Snel, who was an admirer of the work of Ramus. Ramus had no time for Elements X and it is presumably his followers that Willibrord Snel is alluding to here.

[6] Petrus Ramus, *Scholarum mathematicarum libri XXXI,* Basel, 1569 (several later editions), 258. Much of this work is taken up with detailed criticisms of Euclid's *Elements.*

Codrus is a character in Juvenal's *Satires.* Cleopatra was famous for her extravagance, attested, for instance, in Plutarch's *Life of Antony.*

Ramus to understand Euclid: that is, they think that the "Elements" is so called because there is found in Euclid a wealth of every kind of propositions and problems and theorems, for every kind of quantities and of the arts concerned with them, whereas the book is called "Elementary Primer" from its form, because the following proposition always depends on the preceding one right up to the last one of the last Book (and partly also that of the ninth Book), which cannot do without any of the previous ones. Instead of an architect they make him a builders' merchant or a bailiff, thinking that Euclid wrote his book in order to accommodate everybody else, but was the only one who had no home of his own. But that is quite enough on the subject at this point: we must return to the main topic of discourse.

For I saw that the true and genuine distinguishing features of geometrical objects, from which I had to draw out the causes of the harmonic proportions, were totally unknown to the common herd; that Euclid, whose zeal had handed them on, is being hooted off by the scoffing of Ramus, and, as he is drowned by the din of frivolous people, is properly heard by no-one, or is reciting the secrets of philosophy to the deaf; and that Proclus, who could have opened the mind of Euclid, disclosed what was hidden, and made easy what was too difficult to grasp, was being mocked and had not continued his commentaries right up to the tenth Book. I therefore realized that what I had to do completely was, to start with, to transcribe from the tenth Book of Euclid what chiefly related to my present undertaking; also to bring to light the train of thought of that Book, inserting mention of certain definite divisions; and to indicate the reasons why some branches of the divisions were omitted by Euclid. Then, finally, I had to deal with the figures themselves. There, in cases where Euclid's demonstrations were perfectly clear I have been content with a simple reference to the propositions. Many points which were demonstrated by Euclid in another way, had here, on account of the aim which I had in view, that is to say on account of my comparison of knowable and unknowable figures, to be repeated, or linked together if they were separated, or changed in order. I have embraced the series of definitions, propositions, and theorems in continuous numbering, as I did in the *Dioptrice*,[1] for convenience of reference. Also in the actual lemmas I have not been precise, and have not troubled too much about names, as I have been more intent on the matters themselves, seeing that I am now playing the role not of a geometer in philosophy but of a philosopher in this part of geometry. And I wish I could have made my discussion still more popular, provided that it were also clearer and more accessible. But I hope that fair-minded readers will receive my work kindly on both scores, both because I relate geometrical matters in a popular way, and because I could not by diligence overcome the

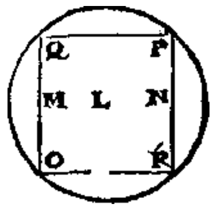
[1] *Dioptrice*, Augsburg, 1611, KGW 4, pp. 327-414.

obscurity of the material. I also give them this final piece of advice, that if they are completely unacquainted with mathematical matters, they should pass over my expositions and read only the propositions, from XXX to the end; and putting confidence in the propositions themselves, without proof, they should pass on to the remaining books, especially the last. They should not be frightened off by the difficulty of the geometrical arguments and deprive themselves of the very great enjoyment of harmonic studies. Now let us proceed to business, with God's help.

ON THE CONSTRUCTION OF REGULAR FIGURES.

I Definition

A plane figure is said to be regular if it has all its sides and all its outward-facing angles equal to one another.



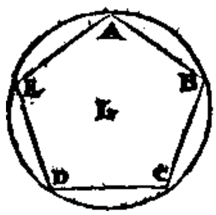
As here in QPRO, the sides QP, PR, RO, OQ are equal and the angles QPR, PRO, ROQ OQP, are equal.

II Definition

Some of these [figures] are primary and basic, not extending beyond their boundaries, and it is to these that the previous definition properly applies: others are augmented, as it were extending beyond their sides, and if two non-neighboring sides of one of the basic figures are produced they meet [to form a vertex of the augmented figure]: these are called Stars.

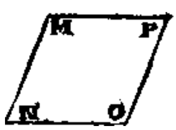
As, here, ABCDE is a perfect five-cornered figure, a primary figure, not requiring any completion which might result from producing its sides.

But FGHIK is a five-cornered star, an augmented figure, constructed by producing pairs of non-neighboring sides, such as AB and DC [produced] to meet at I.



Ill Definition

[Figures] are semiregular if their angles are different from one another but they have four equal sides, like the Rhombi NMPO, GEKD.



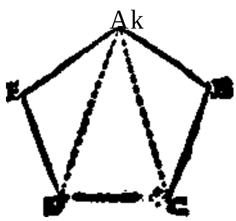
IV Proposition

All regular figures can be placed so that all their angles simultaneously lie on the same circle.

For by Euclid III.21, all equal angles can be inscribed in the same segment, and thus also in equal segments of the same circle, and all the angles of a Regular figure are equal, so all the angles of one figure can be inscribed in equal segments of one circle. But in fact it is necessary that, if one is inscribed, it should be possible to inscribe them all. For all the sides are equal; therefore the segments of the circle cut off by the two sides around one angle are equal, by Euclid III.24. Therefore, as the angle fits, so do the extremities of the sides fit in the same circle. Indeed the extremities of the sides are the angles? It would be otherwise, if although the angles were equal the sides were not equal, for then the necessity that it should be possible to inscribe them all would disappear.

V Definition

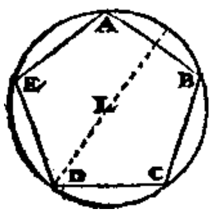
To describe a Figure is to determine by geometrical means the ratio of the lines subtended by the angle to the lines round the angle, and, from what we have determined, construct the Elementary triangles of the figure, and fit the triangles together to complete the figure.



Given the ratio of DA to AE, ED, we form the triangles DAE, DAC, CAB: from which the figure is built up.

VI Definition

To inscribe a Figure in a circle we must by Geometrical means determine the ratio of the side of the figure to the diameter of the circle in which it is to be inscribed, and when we have established this ratio the proposed figure is easily drawn in the circle.



As, if we are given the semidiameter ED, or diameter twice ED, if we know how to obtain from it the correct length for the side DE we can then, by repeatedly taking this length DE round the circumference, easily draw the whole figure.

VII Definition

In geometrical matters, to know is to measure by a known measure, which known measure in our present concern, the inscription of Figures in a circle, is the diameter of the circle.

[8] Throughout this book, Kepler uses "angle" *(angulum)* to mean either the angle enclosed between two straight lines or, as here, the point at which the two sides of a figure meet, namely what today would be called a "vertex" of the figure. However, the context seems too technical to allow "angle" to be translated by the similarly ambiguous "corner."

VIII Definition

A quantity is said to be knowable if it is either itself immediately measurable by the diameter, if is a line; or by its [the diameter's] square if a surface: or the quantity in question is at least formed from quantities such that by some definite geometrical connection, in some series [of operations] however long, they at last depend upon the diameter or its square. The Greek for this is yvcopi^ov, "intelligible."

IX Definition

The construction of a quantity which is either to be described or to be known is its deduction from the diameter, by permitted means, in Greek [these are called] nopina, "practicable."

So construction generally yields either description or knowledge. But Description declares mere quantity, whereas knowledge also in addition declares quality or a definite quantity.[9] *Now a line can be geometrically determined, in Greek TCtKrfi ["fixed"], even though its quality is not yet known intellectually. On the other hand, a line or lines may be known qualitatively, but that does not yet determine them or make them determinate, that is to say if their quality is common to many other things which are different in quantity. So for such lines description is easy, knowledge very difficult. Finally, many things can be described by some Geometrical means or other; but cannot be knowable by their nature: as knowledge has been defined above.*[10]

X Definition

We have proper construction when the number either of the angles of the Figure itself, or of the figure related to it by having either double or half its number of sides, forms the middle term in finding the ratio of the side to the Diameter.

For every regular figure is either itself a triangle or can be resolved into triangles by drawing diagonals. Since, however, every such triangle has its three [angles] equal to two right [angles]; so in the [elementary triangle] of the Trigon

[9] The significance of this remark is made clear by what follows in this and later sections. "Knowledge" consists in being able to construct the required quantity, by the permitted means, and to assign it to one of the categories of quantities set out in *Elements* X. Being "determined" means that the line can be constructed, while being known "qualitatively" means that it is known to what category the line will belong. As Kepler stresses, these two properties are in principle independent. See also Section XII and note 7 below.

[10] A simple example of such a quantity is the circumference of a circle. This can be constructed by the permitted means, that is, it is "determined," but its quantity is not one of the kinds obtainable in straight line form by the permitted means. That is, the circumference cannot be "rectified." In modern algebraic terms, this is expressed by saying that *n,* the ratio of the circumference to the diameter, is a transcendental number.

the angle is one third, in the elementary triangle of the Tetragon the smallest angle is one quarter, in the Pentagon one fifth, in the Heptagon one seventh etc., each fraction being that of two right angles. And it is from the size of the angle that the construction begins.[u]

XI Definition

We have improper construction when the ratio of the side to the diameter cannot immediately be determined Geometrically from the number of the angles, unless the side of another figure is brought in, and this [extra side] is not from the figure with double or half the number of sides [of the original figure].

XII Definition

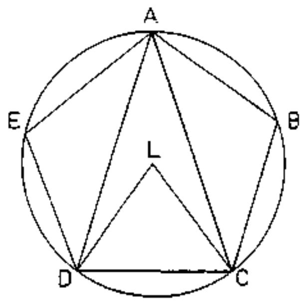
There are various degrees of knowledge, some distant, some close. The first and closest degree is when I know some line and can show that it is equal to the diameter or that a plane surface, which may be formed in another way, is equal to the square of the diameter.[12]

Here the given measure perfectly, that is of itself and by one act, measures the thing that is knowable.

XIII Definition

The second degree [of knowledge] is when if the diameter is divided into a certain definite number of equal parts, or its square is similarly divided, then the line or plane surface we are given is equal to one

[11] The method of resolving a figure into its constituent ("elementary") triangles is shown in Section V above. Kepler apparently regards it as obvious that the vertical angle of the innermost "elementary" triangle of an n-gon will be ^th of two right angles. This may be proved as follows, taking the pentagon as a representative case. (See figure.) Let the figure be inscribed in a circle, center L (as shown in the diagram accompanying section VI above).



Since DC, the side of the pentagon, cuts off one fifth of the circumference of the circle, the angle it subtends at the center will be one fifth of the total angle at the center.

That is,

$$\angle DLC = \tfrac{1}{5} \times 4 \text{ right angles.}$$

Now, the angle any arc subtends at the circumference of a circle is half the angle it subtends at the center (*Elements,* III, 20, Euclid trans. Heath, vol. II, pp. 46-49).

Therefore

$$\angle ADC = \tfrac{1}{2}\angle DLC$$
$$= \tfrac{1}{5} \times 2 \text{ right angles.}$$

[12] That is, the surface is not necessarily square itself but its area is equal to that of a square whose side is the diameter of the circle.

or more such parts. Such a line is called in Greek pr|TT) UTIKEI, expressible in length. Such an area is simply called prjxov, expressible. For number is the medium of expression for Geometers.[13]

We arrive at this degree of knowledge either by description and inscription; or alternatively by its relationship with some other quantity at which we arrive by those means.

And on that account this quality does not determine any quantity; for, as far as I know, it is not sufficient to determine it that I should know something which is compared with it in this way or that for the sake of measurement; I must also know how, that is by what number, it is expressible.™

XIV Definition

The third degree [of knowledge] is when the line is inexpressible in length but its square is Expressible and belongs to the second degree. It is said to be pr|xf| Suvd^ei, "Expressible in square."

XV Definition

Irrational in Latin geometers.

The degrees which follow are all called &A,oyoi, "Inexpressible." Latin translators have rendered this as "Irrational," running a great risk of ambiguity and absurdity. Let us bury this usage, because there are many lines which, although they are Inexpressible, are defined by the best computations.[15] Arithmeticians, by a similar translation, refer to deaf Numbers,[16] that is numbers which cannot speak any more than a deaf man can hear: but under this name they include numbers Expressible only in square as well as inexpressible quantities.[17] Thus

Deaf numbers.

[13] *Numerus enim est Geometrarum sermo.*

[14] What Kepler says here is an extension of what he said in Section IX above (see note 2). Assigning a "quality" to a magnitude, for example by saying that a length is expressible in terms of another 9th (thus being some rational fraction of it), we do not say what the actual length is unless we specify a numerical value for the rational fraction. Kepler's reasons for hammering away at this apparently quite simple point become clear when he is concerned with "unknowable" magnitudes, and numerical approximations to their values, in Sections XLVff. below.

[15]. . . *quia multae sunt lineae, quae quamvis Ineffabiles, optimis tamen continentur rationibus.* "Optimus" here presumably refers to mathematical rigor. Since inexpressible magnitudes (usually called "irrational") contribute to such reasoning ("rationes"), Kepler finds it misleading to give them a name which implies otherwise. Despite his stricture, the usage survives to this day, though the sense of "irrational" has become somewhat more restricted (see note 17 below).

[16] The English name is "surds," from the Latin *surdus* (literally "deaf," as translated here).

[17] In modern usage "rational" denotes numbers expressible in the form ?, where *a* and *b* are integers. Euclid, and Kepler and his contemporaries, use "rational" (and its equivalents) to cover not only these numbers but also numbers whose squares are expressible in this form. Thus the meaning Kepler says arithmeticians give to "surd" (lit. "deaf") is equivalent to the modern "irrational."

the fourth degree in order, and the first in fact of inexpressible quantities, is when neither the line nor its square is Expressible; but nevertheless the Square can be transformed into a Rectangle such that its sides are Expressible at least in square. This line is called Medial,[18] because it is a mean proportional between two expressible lines commensurable only in square: as when one is Expressible in length and the other only in square; or if each is Expressible only in square, but the ratio between the squares is not that of one square number to another.[19]

Such a line is not known or measured by the length of an aliquot part or parts of the diameter, nor is its square [measured by] the square of the diameter; but neither can the two lines between which the Medial is a mean proportional both together be measured by the Diameter; but as for the squares of these lines, these finally can be measured by the square of the diameter.

The square of a Medial [line] is also itself called Medial, whether it takes the form of a square or turns into a Rectangle: so we have this other type of Area, following the Expressible area: And the following kinds [of area] can be distinguished into these two types of area, the Expressible and the Medial.[20]

[18] The word we have translated "medial" is "mese" in the original; that is, Kepler is using a transliteration of Euclid's Greek adjective rather than the usual Latin translation of it. Kepler seems to have worked primarily from the original Greek text of the *Elements* (see note 23 below).

[19] If the ratio between the squares were that of one square number to another, then the original length (taken to be inexpressible) would turn out to be expressible after all.

In algebraic terms, we have a length, /, whose square is equal to the product of two numbers p, q each of which is expressible, at least in square. Thus,

$$I \ x \ I = p \ x \ q \qquad (1).$$

This makes / a mean proportional between p and q, that is, in modern parlance, their geometric mean.

If the ratio between the squares of p and q is that of one square number to another, we have

$$\frac{p^2}{q^2} = \frac{a^2}{b^2} \qquad (2).$$

where a, b are integers.

Therefore, - = -; that is p = - x q, where a, b are integers. Substituting in (1) gives

$$l \times l = \frac{a}{b}(q \times q) \qquad (3).$$

Now, q is expressible in square (at least), therefore the expression on the right of equation (3) is the product of two expressible numbers, and thus is itself expressible. So the left hand side of the equation must also be expressible. That is, / is expressible in square—which contradicts our original assumption.

[20] Compare Euclid, *Elements* X, 21; see Euclid trans. Heath, vol. Ill, p. 49.

XVI Definition

We now come to different individual lines, through the combination of pairs of lines which themselves also introduce new degrees of knowledge. For let us cut either a diameter, or a line commensurable with the diameter at least in square and thus Expressible,[21] or even a Medial line; I say let us cut it into two unequal parts, or let there be made up, from sections of any two such parts of any kind, either by the addition of parts, or by having their squares formed by addition, or subtraction, of such [parts], two lines, I say, that are of different types: they will either be commensurable with one another in length; or, although incommensurable in length, nevertheless commensurable in square. Here, though the individual lines clearly have moved away from commensurability, yet when some of them are combined, either by their squares being put together, or by their being taken as sides to a Rectangle, they make up areas, [such as] those already described, no less than do those [lines] which are commensurable with one another. Since the combination of two such completely incommensurable lines may take many forms, each sinking lower and lower, we shall not be able to assign every pair to a single degree.

XVII Definition

So let the fifth degree of knowledge be when we have two lines which are not both Expressible, nor both Medial, and further are completely incommensurable with one another, and they make both the sum of their squares and their common rectangle an expressible quantity: no less than each of these is made by two lines Expressible in length, by Euclid X.20, or also by two lines expressible only in square, but commensurable with one another in length, by the same [proposition of Euclid].[22] Thus the side of the square [of area] 2 and the side of the square [of area] 8 are in double ratio, because the squares are in the ratio four to one. Thus, although the sides are Inexpressible in length, they are commensurable with one another. Their squares, 2 and 8, add up to 10, an Expressible area. And if they are multiplied one by the other (which is to form [them into] a Rectangle) they make a rectangle of [area] 4, also Expressible. This [i.e. an expressible rectangle] I say is also made by two lines which are neither Expressible nor Medial, and further are completely incommensurable with one another: and for this reason they are not, like the earlier ones, to be assigned to the second or third degree of knowledge, but to the Fifth.

Note therefore that in this degree we shall measure not the lines themselves, nor their individual squares, but instead we shall measure

[21] That is, in modern terms, rational in square.
22 See Euclid trans. Heath, vol. Ill, p. 49.

both the Rectangle formed from them and the sum of their squares; so what is lacking in one square, making it less expressible, is exactly compensated by the other square that is associated with it.

XVIII Definition

The sixth and lower degree of knowledge is when two lines are joined which are neither expressible, nor Medial, both together, and are also incommensurable with one another, and only one of the areas they make is Expressible, while the other is Medial. There are two cases [for lines of this degree]; for either the sum of the squares is expressible and the Rectangle is Medial; or the former is Medial and the latter expressible.

In the former case, the lines are like two expressible lines commensurable only in square. For both the powers, that is the Expressible squares, also have a sum that is Expressible in each case. In fact their rectangle is Medial, by Euclid X, 22 P

In the latter case, the lines are like two Medial lines commensurable only in square, whose ratio to one another is as that of two Expressible lines between which the first of the two Medials is a mean proportional, by Euclid X, 26 and 28.[24] *For because they are commensurable in square: when added the powers give a sum commensurable with the parts [i.e. the powers]. But the parts are Medial, and anything commensurable with a Medial is itself Medial, by Euclid X, 24P*

In this latter case, we are measuring the Rectangle formed by the two lines by the area of the square of the diameter, but we cannot also measure the sum of the squares of the lines: for, for that, we can only find two lines which form a rectangle equal to it, and the squares of these lines we measure by the square of the diameter.

[23] *Sic,* Euclid, *Elements* X, 21 in Heath's translation (see Euclid trans. Heath, vol. Ill, p. 49)—and in Caspar's translation of *Harmonice mundi,* which has similar silent "corrections" elsewhere (for this one see *Weltharmonik,* p. 23).

Most of Kepler's references to individual propositions of the *Elements* give them numbers that correspond to those in Heath's translation (made from Heiberg's Greek text). However, the pattern of agreements and divergences between Kepler's numbering and Heath's indicates that Kepler is referring to the Greek text of the *Elements,* edited by Simon Grynaeus, printed by Johannes Hervagius in Basel in 1533. Grynaeus based his text on Theon's recension of the *Elements.* The volume also included the Greek text of Proclus' *Commentary* on the first book of the Elements.

[24] *Sic,* see previous note. As above, Caspar's translation again contains a silent "correction." Euclid, *Elements* X, 25 and 27 in Heath's translation; see Euclid trans. Heath, vol. Ill, pp. 56, 60.

[25] *Sic,* see note 23 above. Euclid, *Elements* X, 23 in Heath's translation. See Euclid trans. Heath, vol. Ill, p. 53.

XIX Definition

The seventh still lower degree of knowledge is when neither resultant of two mutually incommensurable lines is expressible, neither the sum of their squares nor their Rectangle: but each is however Medial.

In this case, the lines are like two Medials commensurable only in square, one of which is to the other as one of those commensurable lines (that is to say commensurable only in square), between which the Medial truly is a mean proportional, is to some third line, commensurable only in square, by Euclid X, 29.[26]

Euclid is particularly concerned with finding these three pairs of lines, distinguished by making two kinds of area, because they contribute to the composition and structuring of following kinds.[21]

XX Definition

So the eighth degree of knowledge is a continuation of what has gone before, and once more refers to individual lines, but to those which are made up of two terms, namely of two combinations of the preceding combinations, or by the subtraction of one, called the Epharmozusa [conjugate], from the other partner, to make a new kind of line. So that for these we know or measure not complete lines, not the squares of complete lines, not pairs of terms taken one from each, but their combined squares and their Rectangle, as in sections XVIII and XIX above.[28] And although we can enumerate as many degrees of knowledge as there are kinds [of line], so that the earlier degree is always higher than the later: yet because any addition or subtraction refers to its degree, and no operation of addition or subtraction gives rise to diversity, but all are equally related to their pair of Terms or Elements: on this account we shall make them only one degree: but let us recognize that it contains kinds of lines that differ in standing.

XXI Proposition

It is required to know that from two lines commensurable with one another in length nothing can be made which should be taken into account here, whether the lines are Expressible, or Medial, or of lower standing.[29]

[26] *Sic,* see note 23 above. Apparently Euclid, *Elements* X, 28 in Heath's translation, but this proposition does not mention a ratio to a third line. See Euclid trans. Heath, vol. Ill, p. 61.

[27] The pairs concerned are those which Kepler has assigned to the fifth, sixth, and seventh degrees of knowledge. See Sections XVII to XIX above.

[28] This last statement is the geometrical equivalent of the well-known algebraic formula

$$(a \pm b)^2 = a^2 + b^2 \pm 2ab,$$

and is given in Euclid, *Elements,* II.4. (See Euclid trans. Heath, I, pp. 379-382).

[29] That is, we must prove that our definitions cover all possible cases.

For if they are commensurable in length, the whole built up from them will be commensurable with the parts. Now a line commensurable with an Expressible line is Expressible: by the definitions before Euclid X,20.[30] *And a line commensurable with a Medial line is a Medial by 24 of the same.*[31] *And a line commensurable with any of the Inexpressible lines that follow the Medials is of the same kind as it is, by 66, 67, 68, 69, 70, 103, 104, 105, 106, 107 [of Euclid* X/.[32] And so it is also with the other further kinds of line, not mentioned by Euclid, which make more remote degrees [of knowledge]. And even if it were not so for them, it does not matter to us. For they either come down to one of the kinds [of line] which we shall now constitute from incommensurable lines; and thus do not increase the number [of degrees]: or they make lower kinds of their own or another type; and thus they do not belong at this point, where we are setting out the degrees which are closest in rank to those already described.

XXII Definition

So, having dealt with lines commensurable in length, let us go on to those which are commensurable only in square. If two such Expressible lines are combined, they form a Binomial: if they are subtracted the remainder is an Apotome: there are six subordinate kinds of each, see propositions 48 and 85 of Book X [of Euclid].[33]

If we combine two such Medials, which either form an Expressible Rectangle or a Medial one: they will make by addition Bimedials, and by subtraction Medial Apotomes, the former taking their name from the Binomials, the latter from the Apotomes.

Here we may not join up an Expressible line with a Medial one: for two such lines are simply incommensurable, a type that will be discussed in the following section.

[30] In Heath's translation of the *Elements* the only definitions in Book X before Proposition 20 are the four at the very beginning of the Book. (We know from the reference in Section XVII above that Kepler and Heath are in agreement about the numbering of Proposition 20.) These four definitions do not appear to be relevant. However, Prop. 15 (Euclid trans. Heath, vol. Ill, p. 39) appears to be the basis for the first sentence of Kepler's paragraph and the following one may be Kepler's rewording of part of the Lemma which Heath places before Prop. 19 (Euclid trans. Heath, vol. Ill, p. 47) but encloses in brackets as probably being a later interpolation into Euclid's text (ibid., p. 48). See also note 23 above.

[31] *Sic*, see note 23 above. Euclid, *Elements* X, 23 in Heath's translation. See Euclid trans. Heath, vol. Ill, p. 53.

[32] See Euclid, trans. Heath, vol. Ill, pp. 145-151, 229-231, 233-234.

[33] For all six kinds of binomial and apotome, we require Props. 48-53 and 85-90. See Euclid trans. Heath, vol. Ill, pp. 102-112, 178-190.

XXIII Proposition

There remain the lines completely incommensurable with one another. No pair of these can in fact produce the required resultants[34]: as they are both Medial or one is Medial and the other Expressible.

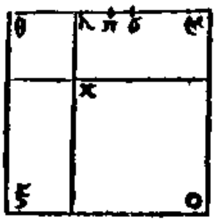
In the one case because the pair is of low standing, in the other because the natures of the two lines of the pair are different. See Euclid X, 71, 108, 109?[b] So no kind of combination can be called in here: we are left only with the lines of lower standing, having excluded the Expressible and the Medial.

XXIV Proposition

From the first pair of such completely incommensurable lines, that is those described in XVII as fifth degree knowable, by adding them or subtracting them there again emerges an Expressible [line]; they are necessarily Binomial and Apotome, see Euclid X, 112, 113, 114.[36] As when both the sum of the squares of a Binomial and of an Apotome, and their Rectangle, is Expressible, it is necessary that the individual Terms of the one should be commensurable with the individual Terms of the other, which is not the case for all Binomials and Apotomes.[37]

Because two such lines which have the two required resultants necessarily form a Binomial and an Apotome; this is proved in the same way as [Euclid] X, 33,™ except that for two pffraiq dvvdusi uovov [lines expressible only in the power] we use pr\xai UT\KEI [lines expressible in length] and for the word usoov [medial] we substitute prfTov [expressible]: and finally we use the definition of the Binomial and the Apotome.

That by the addition and subtraction of a Binomial and an Apotome, with the required resultants, we get back to an Expressible line is seen as follows. For when the sum of the squares is Expressible, and the Rectangle is Expressible; adding the lines together, the square [of the sum] will be composed of the square of each line, and twice the rectangle of the lines, that is, it is composed of two parts which are Expressible: so the whole square will be Expressible: thus so too will be the composite line, whose power is equal to the square. Let the Binomial be Xu, its square KO, and let the apotome be X9, its square OK, and let OK and KO taken together be Expressible, and let the Rectangle made of OX, Xu also be expressible, and two such rectangles KU, KE, complete the whole square of the composite line Ou, which square is Oo.



For subtraction the proof is as follows. If the line composed of OX, uX, that is Ou, is expressible, half of it, On, will also be Expressible (as the larger term)

[34] The word translated "resultants" is *effectus.*
[35] Euclid trans. Heath, vol. Ill, pp. 151, 235-238.
[36] Euclid trans. Heath vol. Ill, pp. 243-254.
[37] Binomial, so *nomen* (here translated "term") for the part.
s« Euclid trans. Heath, vol. Ill, pp. 75-76.

and nX the smaller term, and the other half [of9u, namely] nu; take from it Ha equal to the line OX, and the remainder will also be Expressible, and also the complete line Xa, that is double the line no. But Xa is the remainder after subtracting the Apotome ua from the Binomial Xu. Thus the remainder is Expressible.

XXV Definition

Now from the second pair of the sixth degree (as in section XVIII) consisting of lines completely incommensurable with one another, the sum of whose squares is Expressible, while their Rectangle is Medial; by adding them we obtain a Mizon also called a Major; and by subtraction an Elasson or Minor.[39] From the third [pair], where the sum of the squares is Medial and the Rectangle is Expressible, what we obtain by addition is called the side of a square that is Expressible and Medial, and what we obtain by subtraction is said to Make a Complete Medial with an Expressible.[40] Finally from a fourth pair of the seventh degree (as in section XIX) where each resultant[41] is a Medial; by addition we obtain the side of a square that is Bimedial, and by subtraction [a quantity which is said] to Make with a Medial a Complete Medial.[42]

And here is the Origin of the twelve kinds of quantities treated by Euclid and the explanation for their Number. For Euclid did not consider that he needed to go on to consider more remote kinds [of quantities] which as the sum of their squares, or as their common Rectangle, or both, go beyond the Expressible or the Medial to produce still lower kinds of quantity.[43]

[39] Kepler's "Mizon" corresponds to Euclid's nei^cov in *Elements* X, 39 (Euclid trans. Heath, vol. Ill, pp. 87-88) and his "Elasson" to Euclid's iXdoctov in *Elements* X, 76 (Euclid trans. Heath, vol. Ill, p. 163). Heath prefers to translate Euclid's terms into the Latin equivalents, "Major" and "Minor." Kepler's preference for the Greek terms may be due to the wish to avoid confusion when he considers the greater *(major)* and smaller *(minor)* parts formed by dividing a line, for instance in Sections XXVI, XXVII, and XXVIII below.

[40] That is, when its square is added to an expressible one the result is medial.

[41] The resultants are the sum of the square of the lines and their rectangle.

[42] That is, when its square is added to a medial one the result is medial.

[49] Caspar notes (KGW 6, p. 521) that in the section following *Elements* X, 111 (Euclid trans. Heath, vol. Ill, pp. 242-243) Euclid distinguishes thirteen types of compound irrational lines. However, Kepler seems to be referring to a classification which also includes rational lines. Thus Euclid's total would be fourteen. So Caspar's suggestion (loc. cit.) that Kepler may have omitted medial lines still will not give the correct number. Moreover, since Kepler in fact makes many references to medial lines and areas it seems unlikely that he would omit medial lines from Euclid's list.

In addition, Caspar points out (KGW 6, p. 522) that the properties Kepler notes in this section as those of pairs of lines of the sixth and seventh degrees of knowledge are also properties of pairs of the eighth degree, mentioned in Section XX. This is relatively easily seen once one writes algebraic expressions for the lengths of the lines concerned, which suggests that Kepler never did so.

XXVI Definition and Comparison

These [areas] might suffice us for establishing the degrees of constructions by which the sides of the figures which we need for the study of Harmony are distinguished, if there were not other properties as well as those mentioned, or rather if the properties so far mentioned were not preceded by others that are nobler by which the degrees of knowable constructions are multiplied.

We have come to addition and subtraction; where we have chosen freely the lines that are to be added or subtracted, not imposing any definite quantity upon them. So if we now introduce rules, imposing a definite proportion upon pairs [of lines], not given in such a way that when they were combined they formed one of the twelve kinds [of line already discussed]; but the pairs given in some other way, namely when it is required for one given line also to find its greater part so that the ratio of the smaller to the greater part will be equal to the ratio of the greater part to the sum of them both; or alternatively the ratio of the greater to the smaller will be equal to the ratio of the smaller to the remainder. When the two are subtracted, the result will not always be [a line belonging to] some more remote degree [of commensurability], but in the circumstances, we shall fall back on one of the kinds that have been discussed, and having fallen back on it, we shall compare the line that is found, which of itself is of the eighth degree,[44] with lines of the fourth degree.

For in this way two lines of the fourth degree (as defined in section •XV) together formed an area, from which, when it is cast into the form of a square, there emerged [as the side of the square] a line called a Medial: so the two lines, the Whole and one part, form, by subtraction, the other part, or the two parts, by addition, form the whole. In the former case, the constituent lines were commensurable with one another only in square: in the latter, in place of commensurability, we have identity of proportion between whole and parts. In the former case the similarity of proportion was between the smaller [part] and the line to be formed [i.e. the difference], and between the line to be formed and the greater [part]; in the latter case there is also a similarity of proportion between the two lines to be formed, and between one of them and the proposed whole line [i.e. the sum], for subtraction, while for addition [the proportion is] between one of the lines to be formed, and the proposed line, and the other line to be formed. So, in the former case, given two [lines], a Rectangle was given equal to the square of the line that was to be formed, and thus the area [entered into the question] before the line [i.e. the side of the square]: in the

[44] Lines of the eighth degree of knowledge were described in the previous section (XXV) as being produced by the kinds of procedures described here. However, it is not clear whether Kepler supposes them to constitute a particular category of irrational lines. See previous note.

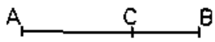
latter case, on the contrary, having made the two [lines] that were to be made, there then follows the equality between the Rectangle of the extreme [lines] and the square of the Mean one, by Euclid VI.17 and II.11.[45]

In the former case the straight lines that make the area had squares commensurable with the square of the proposed Line: in the latter, from Euclid VI.30 we know that we must take a square, commensurable with the square of the proposed line, namely I times it, and from the side of this square we must subtract half the proposed line, so that there remains the required part of the proposed line; and when this part is subtracted from the proposed line there is left the other required part, (or if it is added to the whole we obtain the third required line).[46] And parts that have so many terms should, it seems, be placed in the fourth degree.

[45] Euclid trans. Heath, vol. II, p. 228; Euclid trans. Heath, vol. I, p. 402.

[46] *Elements* VI, 30 (Euclid trans. Heath, vol. II, p. 267) concerns "extreme and mean ratio," that is, the Golden Section. Euclid does not concern himself with actual numerical values, but as Kepler all but does do so it seems appropriate to give an algebraic treatment which will yield such values.

The irrational magnitude which defines the Golden Section is generally designated by the Greek letter *x* (presumably because this is the initial letter of the Greek word for section, TOUT]). Thus, if a line AB is divided at C in the Golden Section we have AC : CB = 1 : x. The defining property of the section is that AC : CB = AB : AC.



Now, AC + CB = AB, therefore AC(1 + *x*) = *d*, therefore

$$AC = \frac{d}{1+\tau} \qquad (1).$$

and

$$CB = \frac{d\tau}{1+\tau} \qquad (2).$$

The relation AB : AC = 1 : *x* gives us AC = xAB. So we have

$$AC = xd \qquad (3).$$

Substituting this value for AC in (1) gives

$$\tau d = \frac{d}{1+\tau}$$

After cancelling the *ds*, multiplying both sides by (1 + t) and taking everything to the left hand side of the equation, we obtain

$$\tau^2 + \tau - 1 = 0 \qquad (4).$$

Using the standard expression for the solutions to a quadratic equation (which is, in its essentials, deduced in the Elements, though not in the neat modern form), we obtain

$$\tau = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} \qquad (5).$$

The negative root shown in (5) is clearly irrelevant in the present case, so we have

$$\tau = \tfrac{1}{2}(\sqrt{5} - 1)$$

That is AC, the larger part of AB that is to be cut off by the Golden Section, is, by (3), $\frac{1}{2}(\sqrt{5} - 1)d$.

Kepler constructs a line of this magnitude by first constructing a square of area $\frac{5}{4}d^2$, finding its side $(= \frac{\sqrt{5}}{2d}d)$, and subtracting \d, which gives him the required line.

The Golden Section has many curious mathematical properties and is associated

For at this point we obtain a line of higher standing than the Medial itself, whichever line it is that the proportion is found in[47]: because the Medial hangs from the proposed Expressible line by a longer chain [of terms] made up of four links: whereas the parts of this [line] depend upon the ratio in which they stand directly to the proposed Expressible line. From this it comes about that there can be many Medials all the same degree away from the Expressible line; indeed the larger part is the one and only part of an expressible line which is in this proportion, and in all cases of any line of lower standing than the Expressible[48] there is one unique part in such proportion. Because of this[49] its construction is equivalent in a sense to the first degree.

Thus when the proposed Straight Line is required to be the whole, and we seek its two parts according to this proportion *(tales),* then Geometers call this division in Extreme and Mean ratio. Certainly this name means that whereas at other times ordinary division of the whole into two parts is not concerned with proportion, or if some line [is constructed] that bears to the whole the ratio of the smaller part to the greater, then there will result four terms, two extremes and two Means: in this case, on the contrary, there are only three terms, the whole and the smaller part being the two extremes; and the greater part being the unique mean term.[50]

It is also, for the same reason, called proportional section.[51] Today both the section, and the proportion it defines, are given the title "Divine," because of the marvelous nature of the section and its multiplicity of interesting properties: the foremost of which is that always when the greater part is added to the whole the compound line is again divided in the same way and the part which was the greater part now becomes the smaller part of the compound line; and the erstwhile whole line becomes the greater part of the compound line, by Euclid XIII.5.[52]

with the regular pentagon (in which it appears, for instance, as the ratio in which the diagonals cut one another). These properties gave it an important place in Renaissance mathematics. Luca Pacioli (ca. 1445-1514), one of the leading mathematics teachers of his day, wrote a treatise about it: *De divina proportione* (Venice, 1509), adding, as the final part of the work, and without reference to its being by a different author, the short book on polyhedra written by Piero della Francesca (ca. 1412-1492). Continuing to exercise his eye for first-rate talent, Pacioli had the illustrations for the book drawn by his friend Leonardo da Vinci (1452-1519) (whose contribution is acknowledged in Pacioli's preface). It seems highly probable that Kepler knew this work.

[47] That is, whichever part it is that defines the ratio of the Golden Section.

[48] That is, a line of lower degree.

[49] That is, the greater part of the line.

[50] See note 38 above.

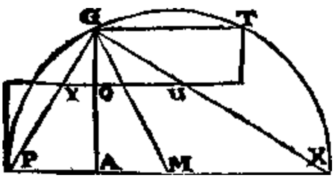
[31] *Sectio proportionalis.* The literal translation "proportional section" is not a term used in modern English.

[52] Let AB be a line divided in the Golden Section at C. Now let AB be produced

XXVII Proposition

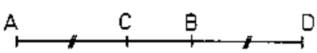
While such a division can be performed on all lines, on a line Expressible in length, on one Expressible only in square, on a Medial line, on a line of one of the twelve kinds we have listed, and on all other kinds of line: in the present work we are concerned only with two of these kinds [of line] which coincide with kinds already explained; according to the two lines which are to be sectioned [in this way]. For it [the line to be sectioned] is either Expressible in length or is a Mizon. If the line which we propose to section is Expressible in length; the greater part of the sectioned line will be an Apotome of the fourth kind; and corresponding to it there will be a Binomial of the same fourth kind, having the same terms as it has.[53] But beware of confusion, for this part is called greater in relation to the proposed line; but the same part is here called an Apotome, not in relation to the proposed line; but qualitatively. If you want to know what it is an apotome of, the answer is that it is an apotome of some line which is commensurable with the proposed line only in square, which namely is the side of a square f times that of the proposed line.[54]

Let GA be the line which is to be divided, and let it be Expressible in length. Construct a right angle GAM and let AM be half the length of GA, and, having joined the points G and M, taking center M and radius GM let there be drawn the semicircular arc PGX, and let AM be produced to cut this arc at P and X, and let there be constructed on the line PA the square PO. Therefore the line GA is divided in proportional section at the point O. So the line AO is the greater part of the line GA that has been divided in proportional section; but the same line AO or the line AP, which is equal to it, is an Apotome not of the line GA but of the line MP or MG, which when square is equal to the sum of the square of GA and of AM, half of GA: so if the square of the line GA were 4 the square of AM would be 1. Thus the square of the line GM would be 5. Insofar as AO or AP is an Apotome it corresponds to the binomial AX: and their common terms are MX, or MP, or MG, and AM.[55]



Now the fact that AP is an Apotome, and AX a Binomial, each of the fourth kind, is proved as follows. For both the terms MX and MA are expressible; how-

to the point D such that BD = AC [see figure]. The property to which Kepler refers here is that C is a point of golden section of AD, AC being the smaller golden section part of AD, while AB is equal to the greater part (i.e. AB = CD). (Though Kepler does not say so, this shows that B is also a point of golden section of AD.) See Euclid trans. Heath, vol. III, p. 448.



[53] However, the terms are added instead of being subtracted.

[54] That is, the area is $\frac{5}{4}$ times the square of the proposed line. See section XXVI and note 46 above.

[55] The terms are subtracted for an apotome, added for a binomial.

ever, they are commensurable only in square, because MX (that is MG) is the side of a square of area 5 units, [the units being] such that MA is the side of a unit square. And the ratio of 1 to 5 is not that of one square number to another. Then the difference of the square 1 and 5 *is 4, a square number whose side, 2, is Expressible in length, and is equal to the proposed line GA. These are the marks of the terms of Binomials of the fourth kind, in the definitions preceding proposition 48, and of Apotomes, in the definitions preceding proposition 85 of Euclid* X.[56]

Lastly if the Expressible line GA is divided in proportional section, its greater part, OA, and the line compounded from OA and AG[bl] *are both of the fifth degree of knowledge. For if their squares are combined their sum is Expressible, namely three times that of the square of the expressible line GA by Euclid XIII.4.™ Indeed their Rectangle is also Expressible, because it is equal to the expressible square of the line GA, since GA is a mean proportional between the part OA and the compound of OA and AG, as was assumed.*

XXVIII Proposition

On the other hand, if any line Expressible in length is thus divided in proportional section, its smaller part will be an Apotome of the first kind.

So if the Expressible line is GA, as before, and when it is divided in proportional section its greater part is AO and its smaller OG; OG will also be an Apotome, by Euclid XIII.6.[59]

Again note that OG is called an Apotome qualitatively, not in relation to the line GA, expressible in length, of which OG is the smaller part; nor in relation to MG, or MP, of which the line AO or AP is an Apotome; but GO has its particular Terms. For since by Euclid X.97[60] *the square on any Apotome, and thus also the square PO, applied to an Expressible line (as here to GT equal to the line GA) produces as breadth GO, an Apotome of the first kind*[m]*: on the other hand, the line AO was an Apotome of the fourth kind. So for the former, GO, the greater term is Expressible in length; and for the latter, AO, the greater term, MP, was expressible only in square. And on the other hand, because the terms are commensurable only in square; it is necessary that the Smaller term, or Prosharmozusa, of the line GO, should be expressible only in square, since for the line AO the smaller term, AM, was expressible in length: however, for both it is true that the difference of the squares of the [individual] terms is a square whose side is expressible in length.*

[56] Euclid trans. Heath, vol. Ill, pp. 101-102 (Definitions II) and p. 177 (Definitions III).

[57] That is, the sum of the lines.

[58] Euclid trans. Heath, vol. Ill, pp. 447-448.

[59] Euclid trans. Heath, vol. Ill, pp. 449-451.

[60] Euclid trans. Heath, vol. Ill, pp. 212-215.

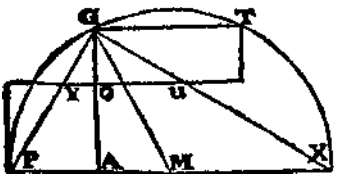
[61] In his translation of the *Elements,* Heath uses the expression "a first apotome," and similarly for other apotomes and binomials.

What are the Terms of GO, a first Apotome, I leave for others to find. Now, as the line GO is a First Apotome, its Prosharmozusa is a single unique line, by Euclid X.79.[62] *This line must be such that its square shall be Expressible, but not by a square number; and the line must, together with GO, make a single line Expressible in length; and by X.30,*[63] *if this one complete line is made the diameter of a circle, say PX; and if the Prosharmozusa, somewhat longer than PA (provided the whole were equal to the line PX) were from one end of the Diameter, X, applied to the circumference^ to give the line XG; then the line joining the points G, P must be commensurable in length with the line PX.*[6i]

XXIX Proposition

Now when a division in proportional section is made on any line that is a Mizon; whose square is equal to the rectangle with length compounded from a given expressible line and the line whose square is five fourths that of the given expressible line, and with breadth whose square is five fourths [of the same square]; then the smaller part will be an Elasson: where Elasson is a term not of comparison but denoting quality: while the greater part will be another Mizon, [the term] again being understood qualitatively, whatever its Elements may be.

As before, let half of the proposed line expressible in length be GA, and half of that again be AM; so that in units such that the square ofGA is 4 the square of AM is 1, and let GAM be a right angle, so in these units the square ofMG will be 5. Let MA be produced in both directions, and with center M and radius MG let there be drawn a semicircular arc PGX. So PX is twice the line GM; therefore the square ofPX will also be five fourths of the square of the proposed line, twice the proposed line GA. But the combined squares ofPG and GX are equal to the square ofPX, therefore they too are five fourths of the square of the proposed expressible line. Now, if you put PG and PX together to make one line; its square will consist of the two squares of PG and GX and two Rectangles contained by PG and GX,[m] *which are equal to two rectangles contained by GA and PX, that is one rectangle contained by the proposed line, double the line GA, and the line PX, two lines which are both expressible but are commensurable only in square: on which account this rectangle will be Medial, by Euclid X.22.*[61] *So since the square of the whole line PGX consists of the expressible square ofPX, and a Medial rectangle, with the same breadth PX: which two, the square of*



62 Euclid trans. Heath, vol. III. pp. 167-168.

63 Euclid trans. Heath, vol. Ill, pp. 68-69.

64 That is, a chord equal in length to the prosharmozusa.

65 By Pythagoras' theorem in triangle PGX.

66 Because of the identity $(PG + PX)^2 = PG^2 + GX^2 + 2.PG.GX$.

67 *Sic,* see note 23 above. Euclid, *Elements* X, 21 in Heath's translation. See Euclid trans. Heath, vol. Ill, p. 49.

*PX and the rectangle contained by twice the line GA and the line PX, are equal to the rectangle which is contained by PX, an expressible line, and the line which is the compound ofPX and twice the line GA, [parts] which are commensurable only in square, of which parts the greater, PX, has a square greater than that of the smaller (twice the line GA) by an amount incommensurable with it [i£. PX] in length (for the square ofPX is 5 in units such that the square of twice the line GA is 4, so the excess, 1, is the square of some line which is incommensurable with the line PX, because the ratio of 1 to 5 is not that of one square number to another) for which reasons the said line, constructed by compounding PX and twice GA, is a Binomial of the fourth kind: since, I say, the square of the whole line PGX is equal to such a rectangle contained by a fourth Binomial and an Expressible: therefore the whole line PGX will be Medial. The Elements which compose it are the parts PG and GX. For because PA is an Apotome and AX a Binomial: therefore they are Incommensurable with one another in length. Indeed, the ratio of PA to AX is the same as the ratio of the square ofPG to the square of GX. Therefore PG and GX are incommensurable with one another in square and thus also simply incommensurable[68]; and the sum of their squares is expressible, in fact equal to the square ofPX[m]: and the rectangle contained by PG and GX is Medial. Therefore by X.39,[10] the line constructed by compounding PG and GX is a Mizon: and by X. 76, subtracting PG from GX, the remainder is an Elasson [Minor].[n] Anyway, the whole line PGX is divided in proportional section at G. For the ratio of PA to AG is equal to the ratio of PG to XG. But PA is [equal to] OA the greater part formed by dividing the line GA in proportional section, because the square of MP is five times the square of the line MA and the Apotome AP is equal to AO by Euclid II. II.[12] Therefore PG is also the greater part formed by dividing the line G in proportional section; and by XIII.*5,[73] *adding PG, the greater part, to GX, the whole line, we obtain a new whole line PGX which is divided in proportional section at the point G; so now PG is the smaller part of this compound line, and GX its greater part. And thus PGX, which is a Mizon, is divided at the same point G both into its Elements, from which it is characterized as a Mizon, and also at the same time into parts in divine proportion.*

I say that these parts produced by proportional section are at the same time both an Elasson and a Mizon.[1] For because AP is a fourth Apotome, therefore [the rectangle] contained by AP, an Apotome, and PX, an Expressible line, has an area equal to that of a square whose side is an Elasson, by Euclid X.94[75]: and because AX is a fourth Binomial, therefore [the rectangle] contained by*

<8 *Simpliciter incommensurabiles inter se.* Kepler means that the lines are incommensurable in length and not merely in square.

69 Using Pythagoras' theorem in triangle PGX.

70 Euclid trans. Heath, vol. HI, pp. 87-88.

71 Euclid trans. Heath, vol. Ill, pp. 163-164.

72 Euclid trans. Heath, vol. I, pp. 402-403.

"Euclid trans. Heath, vol. Ill, pp. 448-449.

74 Kepler has the correct Latinized-Greek accusatives "Elassona" and "Mizona."

'5 Euclid trans. Heath, vol. Ill, pp. 203-206.

this line and PX, an Expressible line, has an area equal to that of a square whose side is a mizon: but the squares ofPG and GX are equal to the rectangles APX, AXP[76] *each to each, therefore PG is an Elasson and GX a Mizon.*

So here we have agreement between the names of the qualities and the names of the proportions. For PG is called the smaller part *(minor),* with respect to the whole line PGX divided in proportional section at G; and it is also called a "minor" line or "minor" Element of the whole line PGX, as this is, qualitatively, a Mizon; finally, in Greek it is called, qualitatively, an Elasson (which corresponds to the Latin "minor"), this with respect to the other two lines, not shown here, from which it is constructed by their being subtracted one from another.[77]

In the same way GX is firstly called the greater part *(major)* of the whole line PGX divided in proportional section; second it is called the "'major'" line or Element of the whole line PGX, as this is, qualitatively, a Mizon in its own right, as also is the whole line PGX in its own right: but the lines which are compounded to construct the Mizon line GX are not shown here.

I believe it was on account of this agreement between division in proportional section and the division of a Mizon into its Elements that these qualitative Terms *(Nomina)* Mizon and Elasson came to be applied to these kinds [of line].

However, here we should take great care not to lose sight of the distinctions between things; proportional section is an absolute proportion, not tied to one particular line, the first to be mentioned, which proposed line is said to be Expressible[78]: now the species[79] Mizon and Elasson are conceived as indicating definite degrees of departure from the first proposed Expressible line. So the division in divine proportion proceeds indefinitely; but the property of Mizon and Elasson does not follow it[80]: in the former (the course of repeated divisions) the part which was a greater part *(major)* becomes, at the next division, a smaller one *(minor);* in the latter, a line which was qualitatively an Elasson, never in any respect becomes a Mizon,[81] nor a Mizon an Elasson. Thus if the Mizon GX is again divided in proportional sec-

'6 That is, $PG^2 = AP \times PX$ and $GX^2 = AX \times XP$.

[77] That is, the names Mizon and "major" apply to the greater part of the line, while the names Elasson and "minor" apply to the smaller one.

[78] That is, the proposed line is *given* as expressible since it is the one which will be used as a measure.

[79] *Species.* This word has been translated as "kinds" in Section XVff above.

Although rigorous mathematical usage insists that one term shall be used in one and only one sense throughout a work, Kepler in fact seems to allow himself a little "elegant variation." This may merely reflect the fact that Book I was (presumably) not written at a single sitting.

[80] That is, when a line is divided in proportional section (golden section, divine proportion) the greater part produced by the section is not always a Mizon, nor the smaller an Elasson.

[81] The literal meaning of "Mizon" is "greater."

tion, its greater part will be equal to the line PG, and will thus qualitatively remain an Elasson; nor can it possibly become qualitatively a Mizon, as it can quantitatively become the greater part *(major)* [in the course of further division]: as long, that is, as GA is proposed as Expressible.

Do you wish to know why, if PGX is qualitatively a Mizon, and GX is also qualitatively a Mizon, then the greater Element of the line GX cannot also be a Mizon in the same way that the greater Element of the Mizon PGX, namely GX, was itself a Mizon? The answer is that although both the lines PGX and GX are Mizons, the result is different in the former case from what it is in the latter. For in the square of PGX we have the whole square of PX and the whole rectangle contained by twice GA and the line P. But in the square of GX, what is involved is [indeed] the square of PX but only half of it, namely the part that is contained by MX and XP, whereas of the rectangle contained by twice GA and the line P, we have only one quarter, namely that contained by AM and PX. So in the latter case the proportion of Medial to Expressible is different from what it was in the former.[82] Our proposition strives to show that there is agreement between the quantitative results of division in divine proportion and the qualitative description of the parts only if the parts are those formed by the first division of the line PGX, depending on its particular proportion between Medial and Expressible [areas]; the proposition does not hold for further divisions.

Note the following contribution to the closeness of the analogy; that just as GX, a Mizon formed by division in divine proportion, makes another greater Mizon, namely PGX, by the addition of PG, which is the greater part obtained by dividing the line GX in divine proportion: so, on the other hand, PG an Elasson of this kind, formed by division in divine proportion, gives PY a smaller Elasson than itself, namely the greater part formed by division of the line PG, or GV, the smaller part formed by division of the line GX: so just as the greatest line, PGX, when divided in divine proportion, decomposes into the Mizon XG and the Elasson GP, so the second Mizon, GX, would decompose into two Elassons XV and VG, which are equal to GP and PY: and so two Elassons would be compounded to form one Mizon; while a Mizon and an Elasson make another Greater Mizon.

XXX Proposition

Individual Prime numbers of sides define individual classes of figures; and figures are counted as belonging to classes which have a number

[82] The definition of a Mizon *(Elements* X, 39, Euclid trans. Heath, vol. Ill, pp. 87-88) involves Elements the sum of whose squares is rational (Kepler's "Expressible") and whose rectangle is medial. It is clear that the proportion between these areas affects the Mizon resulting from the Elements in question.

of sides obtained by repeated doubling of the Prime [that defines the class].[83]

This follows from the definition in section X of this book. For, if all figures such that the numbers of their sides can be obtained by repeated doubling of one given number of sides of one of them have the same form of proper construction: then they all belong to the same Class on account of their construction. Because bisection [of the sides of a figure] does not alter the type or class [to which the figure belongs], when it is associated with individual figures; because of the simplicity and quality of the Parts, both together: for from the individual arcs of the former figure [the process of bisection] makes only two parts, which are equal. But by trisection, or Quinsection, or division into more parts, you cannot avoid either obtaining unequal parts if there are to be only two of them, or many parts, that is more than two, if they are to be equal. Thus in trisecting an arc [of length] 3 it is either cut into 2 and 1, two unequal parts, or into 1, 1, 1, equal parts but many.[M]

The foregoing proposition is proved thus. Constructibility depends on the number of sides [of the figure], by X of this book. Now prime numbers do not have any numerical part [i.e. factor] in common, for unity, which they do have in common, does not determine a form of division and is thus not a numerical part or number.[*'3] *So the demonstrations constructed by means of these numbers [primes] have nothing in common. Therefore the classes determined by individual primes are distinct. The first of these is that which contains the figures (or sort-ojfigures) with these numbers of sides: 2, 4, 8,16, 32, and so on indefinitely: the second has 3, 6, 12, 24, 48, 96, and so on indefinitely: the Third has 5, 10, 20, 40, 80, 160, 320, and so on indefinitely.*[66] *And there are indefinitely many others.*[*"7]

XXXI Proposition

Individual Numbers which are the lowest common multiples of two Primes (excluding two) define individual classes of Figures.

This follows from the definition in section XI of this book. For if such a

[83] Literally, its *(sc.* the class's) prime.

[84] It is presumably because he is thinking of the actual geometrical division of a circle into parts that Kepler uses words such as "trisection," "quinsection," etc. rather elliptically. He clearly intends them to refer to the process of finding even only one of the points that would be required to carry out complete division into the specified number of parts.

[85] Kepler is following ancient Greek precedent in not regarding 1 as a number.

[86] The figure with two sides (a "sort-of" polygon in the sense of having zero area) is now called a "digon." However, one must be wary of attributing twentieth-century mathematical insights to Kepler, who needs the diameter (a digon) as the polygon corresponding to planets' being at opposition (see Book IV below). In fact, in our own day the digon seems to have been invented in a similarly *ad hoc* way—this time to help with the classification of polyhedra (see Coxeter, Longuet-Higgins, and Miller, 1953).

[87] *Aliae injinitae.*

figure does not employ the number of its angles in the construction of its sides: then the form of its construction is different from all the above, and therefore its class is also different. The number two was indeed excluded from producing a new class when multiplied by any Prime: because the bisection of any angle is Geometrical™ and in fact is the process whereby individual classes are each extended indefinitely: if this were not so there would be no classes, but only individual figures. The first [of the classes to which the proposition refers] is 15, 30, 60, 120, 240, 480, etc. multiplying 3 by 5. The second is 21, 42, 84, etc. multiplying 3 by 7. Indefinitely many others follow, as for 5 times 7. Whence we obtain 35, 70, 140, etc.

XXXII Proposition

But both the squares of Prime numbers, except the square of Two, and the products of these squares with another Prime or the square of a Prime also give rise to individual classes distinct from the preceding ones.

Now the square of a Prime number does not make the same class as the Prime [itself], because since the Prime itself makes a new class of figures, those which divide the whole circle," by section XXX of this book: now the same Prime, dividing not all but only a part of the circle will give a completely different construction,[90] *if indeed it is possible [to give one]: since a Part of a circle is very different from the Whole [circle], different that is in kind, and in its absolute configuration: Let us now concern ourselves with this configuration, since it determines the proof of the construction.*

Now, the square of two is again excluded; for the reason that the figure that has twice two angles, that is, the Tetragon, falls into the first class: if the number four is multiplied by a Prime, it [sc. the figures with that number of sides] falls into the class of the Prime, because four is twice two: and every figure with twice the Number of sides belongs to the same [class] as the figure with the original Number of sides.

The first [of the classes to which this proposition refers] contains the figures with 9, 18, 36, 72, 144, 288 sides and so on indefinitely.

The second contains figures with 25,50,100,200, 400, and so on indefinitely.

The third contains 49, 98, and so on indefinitely.

There are indefinitely many other classes derived from squared [primes].

As 27, 54, 108, 216, 432, and so on indefinitely, from 3 and 9.

As 75, 150, 300, and so on indefinitely, from 3 and 25.

As 147, 294, and so on indefinitely, from 3 and 49.

As 45, 90, 180, 360, and so on indefinitely, from 5 and 9.

As 125, 250, 500, 1000, and so on indefinitely, from 5 and 25.

[88] That is, any angle can be bisected using only straight edge and compasses.

[89] That is, the vertices of the figure are found by performing such a division of the whole circle.

[90] That is, it will give rise to a different construction for the vertices of the figure.

As also 225, 450, 900, and so on indefinitely, from 9 and 25, two squares.

There are indefinitely many more classes, from Primes multiplied by squares [of primes], or by squares of Primes multiplied by themselves.[91]

XXXIII Proposition

If from twice the number of angles of the figure you subtract four you will obtain the Numerator of the parts of a right angle which give you the angle of the figure: the Denominator of the parts is the number of angles itself.[92]

So, for the Triangle, twice three is six, subtract 4, which leaves 2. Thus the angle of the Triangle is equal to two thirds of a Right angle. Similarly, for the Icosigon, twice 20 is 40, subtract 4. Thus the angle of the Icosigon is 36 twentieths or 9 fifths of one Right angle. For the angles of each figure are distributed among a number of triangles which is equal to the number of sides of the figure, less two.[93] *But the angles of any triangle add up to two Right angles: therefore the angles of any figure add up to twice as many Right angles as the figure has sides, less four. This number of Right angles is to be divided by the number of angles of the figure, therefore the former number is the denominator and the latter the numerator of the parts of one Right angle [in each angle of the figure].*

XXXIV Proposition

A circle is cut by Geometrical description[94] into two equal parts; and the line bisecting it is known by first degree knowledge: for it is itself the Diameter.

For the basis[95] *for inscribing figures in a circle is to draw a straight line through a specified point, producing it as far as necessary.*

A straight line bisecting the circle is a diameter, that is, drawn through the center, because the greatest of the equal parts into which a circle may be divided is a semicircle, so the line cutting it into two semicircles is the longest, by Euclid III.14, and so it is the diameter, by 15, and by definition.[96]

[91] That is, the fourth powers of prime numbers.

[92] That is, the angle of a regular polygon with n sides is $(2n - 4)/n$ right angles. This formula applies only to convex figures, not to star polygons (which, as is seen in Section II above, Kepler regards as significantly different from convex ones).

[93] This is easily seen from the diagram Kepler supplied in Section V above.

[94] Kepler is using "description" in the technical sense he defined in Section V above, to mean the construction of the required figure.

[95] *principium.*

[96] Euclid trans. Heath, vol. II, pp. 34-37. In Kepler's writings, as in Euclid's, the word translated "line" denotes a segment of a line. This usage is normal among mathematicians of the day. In 1639 Girard Desargues (1591-1661), in his *Rough Draft on Conies,* makes a special point of the fact that in this particular work lines will be considered to extend indefinitely in both directions (see Field and Gray, 1987, p. 70). This usage of the word "line" has now become the norm among mathematicians.

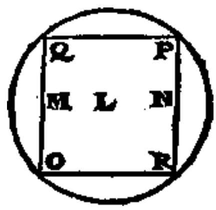
Now the diameter is itself the Expressible line proposed as the Measure of others; it is equal to itself, and its own perfect Measure, the basis[91] of Geometrical knowledge.

XXXV Proposition

The side of the Tetragon derives its Geometrical description [method of drawing] from its angles, if it is drawn independently of a circle *{extra circulum)* and if it is inscribed in a circle this description is of the third degree of knowledge,[98] the description of its square" is of the second degree, and so is that of the area of the figure.

Let the Tetragon be OQPR, its angle, by XXXIII of this book, is a right angle, so by Euclid I.46[100] it is easy, given the side, to describe (draw) the Tetragon.

Since it has four angles it has the same number of sides; thus two sides that meet[101] cut off two quarters of the Circle, that is half the circle. So by XXXIV of this book the end-points of contiguous sides lie on a diameter of the circle. As QO, QP, which form the right angle OQP in the semicircle OQP, have their end-points 0,P lying on the diameter OLP of the circle. So by Euclid 1.47 the [sum of the] squares of the two sides OQ QP is equal to the square of the diameter.[10,2] And if [an area equal to] a half of the square of the diameter is redrawn in the shape of a square, by Euclid 11.14 the side of this square will be the side of the Tetragon.[103] So the square of the side is Expressible.



And because the ratio of the square OP to the square OQ is 2 to 1, not the ratio of a square number to a square Number; [and] OP is in fact Expressible in length: therefore the side OQ is Expressible only in square, by Euclid X.9.[104]

[97] *principium.*

[98] When Euclid discusses the construction of regular polygons, in *Elements* IV, he shows how the various figures are to be inscribed in a circle and how they are to be circumscribed about one, as well as how circles are to be inscribed in the figure and circumscribed about it. The relevant propositions for the square are *Elements* IV, 6 ("In a given circle to inscribe a square"), 7 ("About a given circle to circumscribe a square"), 8 ("In a given square to inscribe a circle"), 9 ("About a given square to circumscribe a circle"); see Euclid trans. Heath, vol. II, pp. 91-95. Our translation of "extra circulum" here as "independently of a circle" is suggested by Kepler's use of the same phrase in Section XXVIII below, where its meaning appears to be unambiguous, see note 127.

[99] That is, the square of the side of the figure.

[109] Euclid trans. Heath, vol. I, pp. 347-348.

[101] That is, sides that are not parallel to one another.

[102] Euclid trans. Heath, vol. I, pp. 349-350. This is Pythagoras' theorem—which Kepler assumed the reader would recognize without prompting in Section XXIX above (see note 61).

[103] Euclid trans. Heath, vol. I, pp. 409-410.

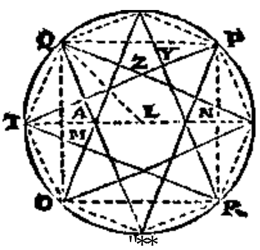
10[4] Euclid trans. Heath, vol. Ill, pp. 28-31.

The area of the tetragon is the same in this figure as the Square of the side, so the area of the Figure is also Expressible.

XXXVI Proposition

The side of the Octagon has a Geometrical description from its Angles, as equally does the side of the star Octagon, being the chord subtending three eighths of a circle, and they [the sides] are of the eighth degree of knowledge, individually, the former being an Elasson, the latter a Mizon; combined, they [the sides] are of the sixth degree [of knowledge], and bear a particular proportion one to another.[105] In short, the area is inexpressible, in fact a Medial.

Let the octagon be UQTOXRSP, and the star UOSQXPTRU[106]*: therefore when a pair of lines, say QT, TO, containing the octagon angle QTO, have a line drawn through their other ends Q O, the connecting line is the side of a Tetragon, because half of eight is four.*[107]



Therefore, after a Tetragon has been constructed in the circle (to leave out other ways of constructing the Octagon), let there be drawn from the center L a perpendicular to its side OQ, to cut the side in M and the arc in T, by using [the construction given in] Euclid I.12.[10S] *Then, by Euclid III.30,*[109] *the two parts of the quarter circle OQ namely the arcs OT, TQ will be equal. Joining the points O and Twill give the line OT as the side of the Octagon, and joining O, S will give OS as the side of the star [octagon].*

Joining the center L with Qj because QML is a right angle, therefore QL [which is] Expressible in length, is, [when] squared, equal to the sum of the squares of QM and ML. Moreover, the semidiameter QL, squared, is equal to twice the square of QM, half the side of the square. Therefore QM and ML are equal, and each is Expressible only in square, by section XXXV above.[110] *Thus the square ofLQ exceeds that ofLM by the square of the line MQ which*

[105] Kepler apparently means that the ratio between the side of the octagon and that of the star octagon has interesting mathematical properties. These are discussed later in the section; see note 109 below.

[106] *Sic.* At first sight, the lettering of the diagram may appear confusing. In fact, the lettering has been designed to make each diagram a development from its predecessors, so that, for instance, PQOR will be a square in all diagrams.

[107] That is, we are going round the octagon taking every second vertex, so that it will take only four steps, instead of eight, to work round the circle.

[108] Euclid trans. Heath, vol. I, pp. 270-271.

[109] Euclid trans. Heath, vol. I, p. 61.

[110] That is, by Pythagoras' theorem in triangle QML we have

$$QL^2 = QM^2 + ML^2 \qquad (1).$$

Also, we know from section XXXV above that the square of the side of the tetragon (QO = 2QM) is equal to half the square of the diameter of the circle (TS = 2QL).

in length is incommensurable with the line LQ But LQ and LS, and LT are equal. Therefore the compound line SM will be a fourth Binomial, whose Terms are SL and LM, by the definitions before Euclid X.48.[1U] *The remainder MT will be a fourth Apotome, whose Terms are TL and LM, by the definition before Euclid X.85.*[u2]

And because MS is a fourth binomial, and ST is expressible, therefore, by Euclid X.57,[U3] *the line QS, which when squared is equal to the Rectangle contained by them,*[U4] *is a Mizon: thus because TM is a fourth Apotome, and TS is expressible: therefore TQ, the side of the Octangle, which when squared is equal to the rectangle contained by MT, TS, is an Elasson, by Euclid X.94.*[iVj]

The elements of these lines are shown in the diagram as PA, the greater one, and AT, the smaller one. For adding AT to PA gives PT, the side of the star: and on the other hand taking TA away from either PA or YT leaves AY, that is QU the side of the Octagon. That is to say, the Elasson TQ, squared, is twice the square of the Prosharmozusa TA; and the side of the Tetragon QP, squared, is equal to the sum of the squares of the elements PA and AQ that is AT.[m]

And the ratio of PX, a Mizon, to the greater of the Elements PA, is the same as the ratio of TQ an Elasson, to the smaller of the Elements TA, and in turn the ratio of the greater of the elements PA to the smaller AT is equal to the ratio of the Mizon PX to the Elasson TQ As the greater is to the smaller; so the sum is to the difference.^[11]

Now these sides SQ QT are not only Mizon and Elasson; but are also lines such that other such lines can be made from them by addition or subtraction. For, first, they are incommensurable with one another, second, the sum of the squares of TQ QS is equal to the square of the expressible line TS. Third, the Rectangle contained by TQ QS is a Medial for it is equal to the rectangle contained by QM, half the side of the Tetragon, expressible only in square, and by TS, expressible in length: from which [it follows] that they [the sides of the

That is $$4QM^2 = \tfrac{1}{2}.4QL^2$$
that is $$QL^2 = 2QM^2 \qquad (2)$$
Therefore, substituting this value for QL in (1), we find
$$QM = ML.$$

[111] Euclid trans. Heath, vol. Ill, pp. 101-102, where this group of definitions is headed Definitions II.

[112] Euclid trans. Heath, vol. Ill, p. 177, where this group of definitions is headed Definitions III.

H3 Euclid trans. Heath, vol. Ill, pp. 125-127.

"[4] The rectangle "contained by" two lines is the rectangle whose sides are the lines in question. Like Euclid, when Kepler refers to two figures being "equal" he means that they enclose equal areas. To convey that two figures are what a present day mathematician would call "congruent," that is, that they are the same in all respects, the figures are said to be "equal and similar."

us Euclid trans. Heath, vol. Ill, pp. 203-206.

H6 AT = AQ. Kepler has not proved this result explicitly but it is implicit in the symmetry of the octagons, each side of the figures being formed from two elements of the same two magnitudes.

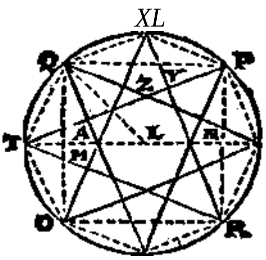
[117] This is presumably the "particular proportion" referred to in the statement of the proposition at the beginning of Section XXXVI (see note 105 above).

two figures] are also of the sixth degree of knowledge when compounded. Therefore by [Euclid] X.39 when they are compounded into one line TQS they make a Mizon,[m] *and by X.76 when TQ that is QZ, is subtracted from QS, the remainder, ZS, is an Elasson.*[119] *Thus it can come about that an Elasson and a Mizon, belonging to one pair, become the Elements of another pair, and the Elasson, subtracted from its Mizon, leaves the Elasson of the other [pair].*

As for the area of the Octagon, the figure is made up of eight triangles like LQT. But the rectangle QTRS is made up of four such [triangles]; therefore it has half the area: and it is a Medial, as just proved; therefore twice this area, that is the area of the Octagon, will also be a Medial, by a porism to Euclid X.24.[12]*? Hence CLAVIUS proves in his Geometria Practica, Book VIII, Proposition 31, that its area is a mean proportional between the area of the inscribed Tetragon and the area of the circumscribed Tetragon, which are in the ratio 1:2, and the [method of] determination of this definite quantity [i.e. that of the area of the octagon] implies that it has the same quality of being a Medial.*[121]

XXXVII Proposition

The sixteen-sided figure *(hekkaedecagon)* has a Geometrical description from its angles, but knowledge of the side takes us far afield into degrees lower than all the preceding ones: and even more so for the sides of its stars, whether they subtend three, five, or seven sixteenths [of the circle].[122]



Because two eights are sixteen, this figure [the 16-sided polygon] can be described by working via the side of the Octagon and thus using the same principles that were employed before in deriving the side of the Octagon via the side of the Tetragon.[123]

Let QO be the side not, as before, of the Tetragon, but of the Octagon and QT, TO now the sides of the sixteen-angle, and let QP be the side of the star Octagon[124]*: before, the first [i.e. QO] was a Mizon: therefore LM, which is half of it, was a Mizon.*

"8 Euclid trans. Heath, vol. Ill, pp. 87-88.

»9 Euclid trans. Heath, vol. Ill, pp. 163-164.

[120] *Sic,* see note 23 above. Euclid, *Elements* X, 23 in Heath's translation. See Euclid trans. Heath, vol. Ill, pp. 53-54. The porism runs "From this it is manifest that an area commensurable with a medial area is medial."

[121] See Clavius, *Geometria practica,* Rome, 1604, Book VIII, Prop. 31 (also called Theorem 13), pp. 409-410. Clavius ascribes the proposition to Oronce Fine (1494-1555) but does not give a detailed reference. He does, however, give a detailed proof of the proposition.

i[22] The stars to which Kepler refers are, in today's notation, {16/3}, {16/5}, and {16/7}. Kepler's diagram shows octagons not 16-gons.

12' In Section XXXVI.

124 There is only one star octagon, namely {8/3}. This is shown in Kepler's diagram as the figure QSOURTPX.

So the rectangle contained by ST, which is Expressible, and LM, which is Mizon, is of a completely new kind, not mentioned among the degrees discussed above, which were of higher kinds. Now a new [area] of this kind, subtracted from that contained by the lines LT, TS, which are Expressible in length, again leaves something of a more distant kind,[125] *namely the rectangle contained by MT, TS which is equal to the square of TQ the side of the 16-sided polygon (Hekkaedecagon). This holds even more for the figures of this Class with more angles, such as 32, 64, 128, and so on.*

Since this holds for one side, the chord subtended by one sixteenth [of the circle], its square, subtracted from that of the diameter of the circle, leaves [the square of] the chord subtended by seven sixteenths [of the circle], so the latter is of more distant degree.

Now the chord subtended by three sixteenths [of the circle] is derived from that subtended by three eighths by bisection so the former is of a more distant degree [i.e. lower degree] than the latter. And the square of the chord subtended by three sixteenths, subtracted from the square of the diameter, leaves the square of the chord subtended by five sixteenths. So this again is of a more distant [i.e. lower] degree.[126]

XXXVIII Proposition

The sides of the Trigon and the Hexagon have a geometrical description, from the angles of the figures; and when they are described in a circle, they are knowable, the former in the third, the latter in the second degree; the surfaces or areas of the figures are Medials, and are in the ratio 1:2.

The construction of a Trigon independent of the circle is very easy, by Euclid 11 127 The most expeditious way of inscribing it in a circle (and I pass over other methods in silence) is to use the side of the Hexagon. Because half of six is three. And the description and inscription of the Hexagon are given in Euclid IV.15.[n]** But it remains to show how the magnitude of the side follows, from the properties (rationibus) of the angles.*

Let the Hexagon be BHCGDF. So since there are 6 angles, the surface of the Hexagon will be divided into six triangles, with their vertices meeting at

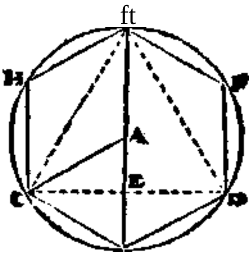
[125] "More distant" means that the quantity is of a still lower degree of knowledge.

[126] The results in the last two paragraphs can be proved in the same way that the analogous results concerning sides of the octagon were proved in the previous section (XXXVI). The final result can be proved by the method described in note 153 below.

[127] Euclid trans. Heath, vol. I, pp. 241-242. Kepler uses the phrase *"Trigoni construct™ extra circulum"* Since *Elements* I, 1 does not refer to constructing a triangle outside a circle, Kepler's "extra circulum" cannot refer to circumscription but must be intended to convey that neither inscription nor circumscription is involved. See also the statement of the proposition in Section XXXV above and note 98.

>[28] Euclid trans. Heath, vol. II, pp. 107-109.

the center A: one of which [triangles] is CAG. Since there are four Right angles surrounding the center A, their sum, divided among the six vertices, gives for the single vertical angle CAG [the value] four sixths or two thirds of a Right angle. But all three angles of triangle CAG add up to two right angles, or six thirds of a Right angle; so subtracting the angle at A, 2 thirds, from the sum of 6 thirds, there remain, for the two angles at C and G, the sum of 4 thirds: now, all the angles are equal; so for each of the angles at C and at G there remain 2 thirds of a Right angle, no less than for the vertical angle at A. But if the three angles are equal, the three sides of the Triangle must be equal also. So CG, which is at the same time the side of the Hexagon and of the triangle that is one sixth of it, is equal to the semidiameter of the circle, CA or AG. Thus the side of the Hexagon is expressible in length, namely half the diameter [of the circle]. So the former belongs to Degree II by XIII of this book.



Let us consider a triangle [inscribed in the circle] such as triangle BCD. Its side BC links two sides of the hexagon CH, HB, which meet at H. Thus since BHC is two thirds of the semicircle and CG one third, therefore the arc BCG is a semicircle, and BG a diameter, passing through A. Therefore the angle BCG, the angle in it [sc. the semicircle] is a Right angle, by Euclid III.31.[129] So [the sum of] the squares of BC and CG is equal to the square of BG, by Euclid I.47.[130] But CG is a semidiameter, and its square is a quarter of that of the diameter; so subtracting a quarter from the square of the diameter BG gives as remainder the square of the side of the triangle BC. So this square is Expressible: but because its ratio to the square of BG is not equal to that of a square number to a square number, but is as 3 to 4, therefore BC is expressible only in square. It therefore belongs to the Third degree, by section XIV above.

And because BC, BD are equal, and the angles BCD, BDC are equal; therefore BE, the perpendicular dropped [from B] to CD, will cut it in E into equal lines CE, ED. The complete line CD was Expressible only in square; so the same is true of half of it, CE. Therefore the rectangle contained by CE, AG, lines commensurable only in square, the latter being Expressible in length, is a Medial. But this Rectangle is equal to the [sum of the] areas of two triangles, each equal to the triangle CGA (one of the six triangles that make up the Hexagon),[131]

[129] Euclid trans. Heath, vol. II, pp. 61-63.

[130] Euclid trans. Heath, vol. I, pp. 349-350. This is Pythagoras' theorem, which Kepler sometimes uses without giving a reference (e.g. in Section XXIX above, see notes 69 and 102).

[131] It may be merely an oversight that Kepler does not cite Euclidean chapter and verse for this argument. However, he is in fact departing somewhat from the mathematical style of the *Elements.* Euclid deals with areas of parallelograms and triangles, and their relationships, in *Elements* I, 33-45. Propositions 33 to 36 concern parallelograms, 37 to 40, triangles. Proposition 41, which Kepler uses here, states that "If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle" (Euclid trans. Heath, vol. I, p. 41). However, the parallelogram Kepler considers is actually a rectangle, so his statement seems

and is thus one third of the Area of the Hexagon. So the area of the Hexagon is a Medial surface. And because the triangles BCA and BCH have sides BA and BH, CA and CH equal, and one common side BC: they therefore have equal areas.[vi2] *But BCH, BDF, CDG are parts of the Hexagonal area, [the parts] by which it exceeds the Triangular area BCD, which is equal to the sum of the triangles BAC, CAD, DAB. Therefore the Hexagonal area is twice the Triangular area. Thus the Triangular area is also Medial, because it is commensurable with, that is twice, the Hexagonal [area], which was Medial.*

XXXIX Proposition

The sides of the Dodecagon and of the star of the same name, namely the chords subtended by five twelfths of the circle, can be described Geometrically, and when they are inscribed in the same circle they [sc. the sides] are individually knowable in the eighth degree of nobility of knowledge *(nobilioris cognitionis),* taken together [they are knowable] in the fifth degree; in fact the surface of the Dodecagon is Expressible.[133]

Let the dodecagon be BMHLCKGQDPFN and the star Dodecagon BKFLDMGNCPHQB.

So, because twice six are twelve, these figures can be described by working via the side of the Hexagon, using the same principles as were employed before in deriving the side of the Octagon via the side of the Tetragon,[VM] *[namely by] drawing from A, the center of the circle, a line perpendicular to HC, the side of the hexagon, to cut the side in 0 and the circle in L and P, and joining the points L, Hfor the side of the Dodecagon, and the points H, Pfor the side of the star.*

So since HC, the side of the hexagon (sexanguli) is Expressible in length; so too will be its half HO, but AC, equal to HC itself, is, when squared, equal to the square of its half OC plus the square of 'AQ[135] *therefore the ratio of the*

rather to refer to the determination of the area of the triangle, which does not occur in the form of a single proposition in the *Elements,* but is, of course, a standard mensuration problem found in many mathematical textbooks:

Area = \ base x height.

[132] In modern parlance, the triangles are congruent. Kepler uses "congruent" in a different sense; see Book II below.

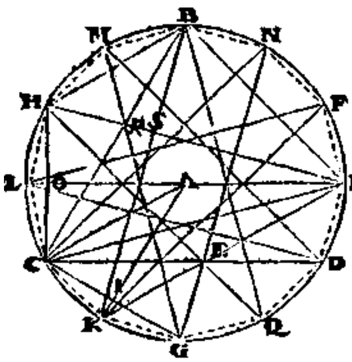
[133] The star dodecagon to which Kepler refers is {12/5}. He presumably regards this as equivalent to {12/7}, which is that same figure but is described in the opposite direction round the circle.

Kepler assumes the star {m/n} is the same as {m/(m - n)} for all the polygons he considers (see earlier sections). He is either not interested in problems of chirality or is unaware of them, though he must have known Proclus' account of Pappus' proof of the equality of the base angles of an isosceles triangle by considering the triangle with its vertices read clockwise to be distinguishable from the same triangle with its vertices read anticlockwise (Proclus, *Commentary on the First Book of Euclid's Elements,* 249-250, see Proclus trans. Morrow, 1970, pp. 194-195). Kepler's neglect of chirality affects the number of distinct tessellations he recognizes in Book II (see below, Book II, Section XIX and note 18).

[134] See Section XXXVI above.

[135] Using Pythagoras' theorem in triangle AOC.

square of AO to the square of AC or AP is that of 3 to 4, not the ratio of a square number to a square number.[136] *Therefore PA, AO are commensurable with one another only in square, as are LA andAO. And CA, that is PA or AL, the greater, which is Expressible is, in square, greater than the square of the smaller, AO, by [the square of] a magnitude which is commensurable with CO itself. Therefore, by the definitions before Euclid X.48,*[i37] *the compound line PO is a Binomial, and by the definition before [Euclid X.]85,*[13H] *the remainder OL is an Apotome, each designated First.*[139] *The Terms are AP, Expressible in length, andAO, Expressible only in square. But, by Euclid X.54,*[]40] *HP, in square equal to the rectangle contained by OP, a first Binomial, and PL, which is Expressible, is a Binomial, and by 91 of the same,*[141] *the side of the Dodecagon, in square equal to the rectangle contained by OL, a first Apotome and LP, which is expressible, is an Apotome. Thus they [the sides] belong in the eighth degree of nobility of knowledge.*[142]



The Terms of this compound line PH, and of the diminished line HL, are PS and SH.[U3] *And since HB is the side of the Hexagon* (sexanguli), *KP of the Triangle, [and] BP of the tetragon* (quadranguli), *the square of the first is equal to twice the square of the smaller Term, that is it is equal to the square of HS plus [that of] SB, the square of the second is equal to twice the square of the greater [Term], that is it is equal to the square of KS plus [that of] SP;*

[136] That is, in algebraic terms,

$$AC^2 = \left(\frac{AC}{2}\right)^2 + AO^2.$$

Therefore

$$\tfrac{3}{4}AC^2 = AO^2,$$

$$\frac{AO^2}{AC^2} = \frac{3}{4}.$$

[137] Euclid trans. Heath, vol. Ill, pp. 101-102. where this group of definitions is headed Definitions II.

[138] Euclid trans. Heath, vol. Ill, p. 177, where this group of definitions is headed Definitions III.

[139] That is, a First Binomial (Definitions II, 1) or a First Apotome (Definitions III, 1). For references to the definitions, see notes 137 and 138 above.

[140] Euclid trans. Heath, vol. Ill, pp. 116-119.

[141] That is, *Elements* X, 91, Euclid trans. Heath, vol. Ill, pp. 190-193.

[142] Kepler appears to be consistent in the words he uses for establishing rank: "of higher degree" is equivalent to "more noble"—and the first degree is, of course, higher than the second, and so on. These terms establish degrees of knowability. These degrees are different from the "species" to which a line belongs (such as "medial" or "binomial"), which are said to be of different "standing." In *Elements* X, Euclid is concerned only with this second form of classification, in which, in his version, the notion of higher and lower types is not explicit.

[143] That is,

$$PH = PS + SH$$

and

$$HL = PS - SH.$$

while the square of the last is equal to the sum of the square of both [Terms], each taken once, that is [the squares] of BS and SP.[144]

PH is also a binomial composed of PR, the side of the square, and RH, the side of the Dodecagon; but it is not called a Binomial on account of this composition; because, by Euclid X.42,[145] *there is only one point, here the point S, which can divide it into its Terms.*

And since HO, LP are Expressible in length, the rectangle they contain, that is the rectangle contained by LH, HP will be Expressible,[146] *and the sum of the squares of LH and HP is similarly Expressible, being indeed equal to the square of LP itself*[H1] *Therefore on this basis LH and HP taken together are in the fifth degree of knowledge. Neither do they produce anything new*[148] *when they are combined, nor do they produce a Binomial or an Apotome again; for adding LH and HP gives a line Expressible only in square, that is the line whose square is three halves of the square of LP: while subtracting LH, or HR, from HP again produces a line Expressible in square, [namely] PR, the side of a square, whose square is half the square of LP.*[U9]

144 p o r the side of the hexagon, HB, we have, by Pythagoras' theorem in triangle HSB, $HB^2 = HS^2 + SB^2$.
But $SB = HS$,
therefore j-jg2 _ $2HS^2$.

Thus the square of the hexagon side is twice the square of the smaller Term.

For the side of the triangle, KP, we have, by Pythagoras' theorem in triangle KSP,

$$KP^2 = KS^2 + SP^2.$$

But $KS = SP$,
therefore $KP^2 = 2SP^2$.

Thus the square of the side of the triangle is twice the square of the greater Term.

For the side of the Tetragon, BP, we have, by Pythagoras' theorem in triangle BSP,

$$NP^2 = BS^2 + SP^2.$$

Thus the square of the side of the tetragon is equal to the sum of the squares of the two Terms.

145 Euclid trans. Heath, vol. Ill, pp. 92-93.

146 Since HO is perpendicular to LP, the area of the rectangle contained by HO and LP will be equal to twice the area of triangle HLP.

Since LP is a diameter, angle LHP is an angle in a semicircle and is therefore a right angle. So the area of the rectangle contained by LH and HP will be equal to twice the area of the triangle HLP.

That is, the rectangles contained by HO, LP and by LH, HP will be equal to one another, as Kepler says.

An argument similar to that given here is implicit in what Kepler has written. On the difference between this mathematical style and that of *Elements* I, see note 123 on section XXXVIII above.

147 By Pythagoras' theorem in triangle LHP.

148 "New" must mean "of a new species," as it does below.

149 Kepler has left the proof of these results as an exercise to the reader. If we allow the use of algebraic expressions (which in the present case could be recast in geometrical terms) there is a simple proof as follows.

By the well known identity,

$$(LH + HP)^2 = LH^2 + 2LH.HP + HP^2 \qquad (1).$$

(continued)

And since the Area of the Dodecagon is made up of 12 Triangles, one of which is LAC, four of these [the triangles] might be contained in the Expressible rectangle LHPD, that is [it has] a Third of the total Area,[150] *therefore the total Area is also Expressible, namely of quantity equal to three times the product of HO and LP; so the Area is Three Quarters of the square of the diameter, or the Arithmetic Mean between the Tetragon circumscribing the circle and the Tetragon inscribed in the circle; just as the Area of the Octagon (Octanguli) is the Geometric Mean between them.*

XL Proposition

The Regular figure with twenty-four sides, and all figures obtainable from it by repeated doubling of the number of sides, can be inscribed [in the circle] Geometrically, but the knowledge of the side strays further into degrees still more distant from those considered earlier: and the same applies to the sides of its stars, which subtend 5, 7, or 11 twenty-fourths [of the circle].[151]

This is proved as Proposition XXXVII above was from the sixteen-sided figure; but with this difference, that here the side of the star Dodecagon and

Now, LP is a diameter, therefore angle LHP is a right angle, and by Pythagoras' theorem in triangle LHP we have

$$LH^2 + HP^2 = LP^2 \qquad (2).$$

Also, LH.HP, the product of the base of the triangle LHP and its height, is equal to twice the area of the triangle, which (as proved in note 138 above) is equal to $\frac{1}{2}$.HO.LP. So we have

$$LH.HP = HO.LP \qquad (3).$$

Substituting from equations (2) and (3) into (1) gives

$$(LH + HP)^2 = LP^2 + 2HO.LP \qquad (4).$$

Now, $2.HO = HC$

$= \frac{1}{2}LP$ (since HC is the side of the inscribed scribed hexagon, i.e. equal to the radius of the circle). Therefore, the second term on the right hand side of (4) becomes $\frac{1}{2}LP^2$ and the equation simplifies to

$$(LH + HP)^2 = \tfrac{3}{2}LP^2.$$

This is what Kepler required.

$(LH - HP)^2$ can be evaluated in a similar manner, to give an expression like that on the right hand side of (4) but with the second term subtracted instead of added, so that the right hand side eventually becomes $\frac{1}{2}LP^2$.

150 The proof has again been left as an exercise to the reader.
Since CO is perpendicular to LP,

Area of triangle $LAC = \frac{1}{2}.LA.CO.$

But $CO = HO$ and $LA = \frac{1}{2}.LP,$

therefore Area $LAC = \frac{1}{4}.LP.HO.$

Now, it has already been proved (see note 138 above) that

$$LP.HO = LH.HP.$$

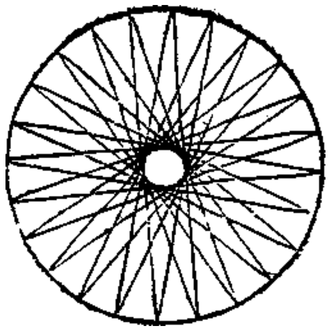
Therefore, Area $LAC = \frac{1}{4}.LH.HP$

$= \frac{1}{4}$Area LHPD, as required.

151 The stars are the polygons {24/5}, {24/7}, and {24/11}.

half of it are first Binomials, so that the rectangle contained by the half [side] and the diameter, [the latter] being Expressible, this time does not yet come to be of a new species, because by 54, the side of the area is again a Binomial.[152] *But now this rectangle, subtracted from the Expressible [rectangle contained by] the complete [diameter] and the half Diameter, leaves a new species, of which no mention has so far been made, and of lower degree, because more composite; and this is produced by the square of the side of the figure with 24 angles.*

This is all the more true for figures of this Class with more angles; such as figures with 48 angles, 96 angles and so on.



The chord subtended by five twenty-fourths of the circle is disclosed by bisection of the arc containing five twelfths: the square of the former [i.e. -ii] subtracted from the square of the diameter, leaves the square of the chord subtended by seven twenty fourths[153]*: so the square of the side, or the chord subtended by one twenty-fourth, in the same way forms the square of the chord subtended by eleven such parts.*[154] *So they all belong to a more distant degree.*

XLI Proposition

The side of the Decagon and that of the star decagon, or the chord subtended by three tenths of the circle,[155] have a Geometrical description through their angles, and can be inscribed in the circle; and they are knowable, separately as individuals indeed in the eighth Degree of knowledge, while combined [they are of] the fifth degree; and combined with the semidiameter they are of the fourth degree.

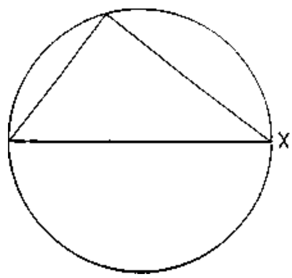
Let the Decagon be BCDEFGHIKL, and its star BEHLDGKCFIB. Therefore, since there are ten angles, the surface of the figure will be composed of ten triangles meeting at the center A, one of which [triangles] is FAG. So dividing up the sum of four right angles, which surrounds the point A, into the ten

[152] *quia potens Mam per 54. est iterum Binomis.* The "side" means the side of a square whose area would be equal to that of the rectangle in question. This usage is found in Heath's translation of Euclid.

Kepler's elliptical reference is to *Elements* X, 54, Euclid trans. Heath, vol. Ill, pp. 116-119.

[153] This can be proved as follows.

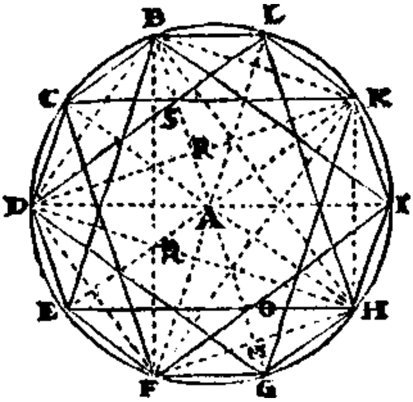
Let X, Y, Z all be vertices of a regular 24-gon inscribed in a circle, such that X, Y are opposite vertices and Y and Z are separated by five sides of the 24-gon (i.e. YZ subtends ^ths of the circle). Since XY is a diameter, the angle YZX is a right angle. By Pythagoras' theorem in triangle XYZ we have $XY^2 = YZ^2 + ZX^2$. Therefore $XY^2 - YZ^2 = ZX^2$, which is the result Kepler requires.



[154] This can be proved by using the method of note 153, but making Z the vertex next to Y.

[155] The star decagon is {10/3}.

vertical angles of these triangles, each one comes to 4 tenths or 2 fifths of one right angle. But the sum of the three angles of this triangle is 10 fifths, that is 2 Right angles; therefore taking from this the vertical angle at A of two fifths, the remainder left for the two base angles is 8fifths: and since they [the base angles] are equal, each of them is of [size] 4 fifths. Thus each of the base angles is twice the angle at the vertex. The proof that follows hangs on this [result].



For if the angle AFG is divided into two equal parts by the line FO, using Euclid 7.9[156]; *the angles AFO, OFG will be equal to one another, and each will be 2 fifths of the Right angle; thus each will be equal to the angle FAO. So by Euclid VI.3,*[151] *the ratio of AF to FG will be equal to the ratio of AO to OG.*

Now because [angle] OFG is 2 fifths [of a right angle], and the angle OGF (namely AGF) was 4 fifths, therefore [angle] FOG will also be 4 fifths. So the angles at O and G being equal, the sides opposite them, FG, FO will also be equal.

In the same way, in the triangle AOF also, since [angle] AFO is 2 fifths [of a right angle], as was angle FAO; therefore AO and FO (that is the side [of the decagon] FG) will be equal to one another. Now, the ratio of AF to FG is equal to the ratio of AO to OG, as already proved; therefore, also, the ratio of AG to its part, AO, is equal to the ratio of the latter to the remainder, OG. Thus the Leg[158] *AG is divided proportionately in the point* O.[159] *So, by Euclid XIII.5,*[160] *if OA or OF, is produced to I,*[161] *so that 01 is equal to the whole line AG, FI is also divided proportionately at O, and if the points A and I are joined, AIO will be a triangle congruent (congruum) with the initial triangle FAG, so that [the angle] OAI will be twice [the angle] FAO, and [angle] FAI will be 6 fifths [of a right angle]. Accordingly, if with center A, and compass opening AG, the circle FGI is drawn, FG will be the side of the decagon, the greater part [produced when] the semidiameter is divided proportionately, and FI the side of the star, or the chord subtended by three tenths [of the circle], is the line composed of [i.e. the sum of] FO and 01, [that is] the side of the decagon and the semidiameter.*

On account of this, these sides, taken together with the semidiameter, can be accepted as belonging to the fourth degree, by XXVI above.

And since the line that is divided [in proportional section], AG, is Express-

[156] Euclid trans. Heath, vol. I, p. 264.

[157] Euclid trans. Heath, vol. II, pp. 195-197.

[138] *Crus*, corresponding to the literal meaning of isosceles as "equal legged."

[159] That is O divides AG in what Euclid calls "mean and extreme ratio," also known as "divine proportion" and "golden section."

[160] Euclid trans. Heath, vol. Ill, pp. 448-449.

[161] The point called I in Kepler's text is shown marked J in his diagrams.

ible in length, and the side of the decagon is the larger part [produced by the division]; and the side of the star is composed of[i.e. the sum of] the whole [line AG] and its greater part; accordingly, by XXVII above, the former [the side of the decagon] is an Apotome and the latter [the side of the star] is a Binomial, each of the fourth kind[162] *in this respect they belong to the eighth degree of knowledge, closely following the side of the Dodecagon and its star, and exactly on a level with the side of the Octagon and its star.*

And by XXIIX above,[163] *the remainder OG also, and also half of it, NG, is an Apotome of the first kind. But beware of supposing that AG is its greater term and AN its smaller one.*

Finally, by the same XXVII above, the sides GF, or OF, and FI combined not with the semidiameter but with each other belong to the fifth degree of knowledge, because both the sum of their squares and their common Rectangle are Expressible.

So adding together the side of the Decagon and the side of its star gives a line expressible only in square, its square being equal to five fourths of the square of the semidiameter, which [i.e. this line] in the earlier figures from Prop. XXVII is [shown as] PX, composed of PA (equal to the line OA) and AX: between which there is the mean proportional GA which is expressible.[164]

On the other hand, subtracting the side of the Decagon, OF, from the side of the star, FI, leaves the Expressible line OI, that is the semidiameter.[16‚3] *So this gives nothing new.*

XLII Proposition

The sides of the Pentagon and the Star Pentagon, or the chord subtended by two fifths of the Circle,[166] have a Geometrical description through their angles, and are knowable, separately in the eighth degree; combined, in the sixth and in the fourth degree of knowledge.

Description independently of the circle proceeds thus: if the proposed (futurum) side is given in length, we shall divide it in proportional section by Euclid II. 11 or VI.30,[]67] *and to it we shall join the greater part formed by the section: and having drawn two sides (crura) each equal to this composite line; and making the proposed line the Base, we shall construct the interior triangle of the*

As here FB
BH legs (cn *tra*)
FH base.

[162] For Euclid's classification of binomials and apotomes in *Elements* X, see Euclid trans. Heath, vol. **Ill,** pp. 102-103 and 177.

[163] *Sic.* It is number **XXVIII.**

[164] Kepler has compounded the side of the decagon and the side of its star, thereby obtaining a line expressible only in square. Its square is equal to f times the square of the semidiameter of the circle. The compound line (the sum) is shown in the diagram accompanying Prop. XXVII (but not in those for Props XLI, XXIX, or XXVII), where it corresponds to PX, whose square is f times the square of twice GA. In this diagram, between PA and AX there is the proportional mean GA which is expressible.

[165] Earlier in this proposition Kepler has shown that the triangle AOI is isosceles, its two "legs" being AI, IO (see the passage following the reference to note 158 above).

use j n estar pentagon is {5/2}.

‚67 Euclid trans. Heath, vol. I, pp. 402-403; ibid., vol. II, pp. 267-268.

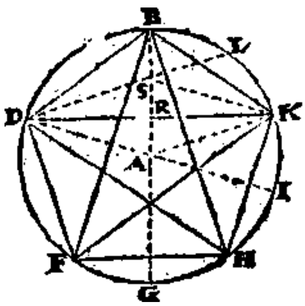
Pentagon.™ Now, since the composite side (crus) comprises the whole proposed line and the greater part found by dividing it in the divine section; the composite line will also be divided in this way [i£. in proportional section], and its greater part will be the proposed side, so that the base angle of this Triangle will be twice the angle at the vertex, as above for the Decagon[169]: and on the two said equal sides (crura) of the triangle, serving as bases, we shall add two triangles to the outside [of the figure], the triangles having their [remaining] sides (crura) equal to the proposed side.

As here FDB on FB and BKH on BH.

The easiest inscription in the circle is via the side of the Decagon. For, since half of ten is five, if we join up the ends F, H of two sides FG, GH of the Decagon that meet at G, the line FH will be the side of the Pentagon, and similarly for HK[110]; and if we join up the ends F, K, the line FK will be the side of the star. So let the Pentagon be BDFHK and its star BFKDHB.

Euclid, then, shows in XIII.10 that the square ofFH, the side of the Pentagon, is equal to the sum of the square of FA, the side of the Hexagon, and of the square of FG, the side of the Decagon, that is [the sum of the squares of] the semidiameter, AG, and GO, the Greater part resulting from the [Proportional] section [of the semidiameter].[171] This proof in Euclid is somewhat difficult to understand; so I shall try to give an easier one here.

From the ends of the side of the Pentagon B, D let there be drawn through the center A the straight lines BG and DI: and as DB subtends two tenths [of the circle], similarly let the neighboring line DL subtend three [tenths] and DKfour [tenths], these lines cutting BG in the points S and R [respectively].[172] So the angle LDI, that is SDA, is two fifths of a Right angle, because [the arc] LI is one fifth part of the circle, just as FH also, and indeed arcs equal to it, subtend equal angles at the circumference, by Euclid III.21 or 27.[m] Indeed, angle DAB, that is angle DAS, is equal to four fifths of a right angle, because DB is a fifth part of the circle, whose complete circumference marks out four right angles at A. So the sum of angles SAD and ADS is six fifths of a right angle. But all three [angles of the triangle SAD] add up to ten fifths. Therefore the remaining angle, DSA, is four fifths. So [angle] DSA is equal to the [angle] DAS, and the side DS is thus equal to the side DA, which is a semidiameter. Therefore,



Put O where DH, FK, AG cut each other.

[168] Compare Section V above, with its accompanying diagram, and *Elements* IV, 10 and 11, Euclid trans. Heath, vol. II, pp. 96-102.

[169] That is, for the constituent (elementary) triangles of the decagon. See the first paragraph of the proof in Section XLI above.

[170] See diagram in Section XLI above.

i" Euclid trans. Heath, vol. Ill, pp. 457-460.

[172] The letter O has been added in the diagram supplied in KGW 6, p. 44. In Kepler's diagram, DL is shown as a broken line, to indicate that it is constructed. It has been constructed to subtend three tenths of the circle, as in Section XLI above, where DL is shown in the accompanying diagram.

™ Euclid trans. Heath, vol. II, pp. 49 and 58-59.

by the above, the greater part obtained by proportional section of the semidiameter DA is equal to SA, so SA is equal to the side of the Decagon, by what has [already] been said.[174] *And DA is the semidiameter, that is the side of the Hexagon. So I say that the side of the Pentagon, DB, squared, is equal to the sum of the squares of SA and* AD.[175]

For, joining K to S and to A, since DA, AK are equal, and DS, SK are equal to them, the parts SR, RA will also be equal, and angle DRB is a right angle. Therefore DB squared is equal to the sum of DR squared and RB squared.[116] *But DR squared is less than DA squared, by the amount RA squared,*[177] *and BR squared is less than BA squared by an amount which is the sum of the rectangle contained by BR, RA, taken twice, and the square of the line RA.*[m] *So the sum of the squares of DR and RB is less than the sum of the squares of DA and AB by twice the rectangle contained by SA, AB, that is by the rectangle RA, AB taken once™ But the two rectangles contained by SA, AB and SB, BA together make up the whole square of BA.*[m)] *Therefore, on subtracting the rectangle contained by SA, AB, there remains the square of the line DA, plus the rectangle contained by SB, BA, and together they are equal to the square of DB.*[m] *Now, since the semidiameter, BA, is divided in proportional section at S, and the greater part is AS: so the rectangle SB, BA is equal to the square of SA.*[1*2] *Therefore the side of the Pentagon, squared, is equal to the [sum of the] two squares of DA and AS; that is the squares of the sides of the Hexagon and the Decagon.*[183,]

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171 In Section XLI above.

175 That is, what Kepler has to prove can be reduced to this form.

m That is, $$DB^2 = DR^2 + RB^2 \qquad (1)$$
by Pythagoras' theorem in triangle DRB.

177 That is, $$DR^2 = DA^2 - RA^2 \qquad (2)$$
by Pythagoras' theorem in triangle DRA.

178 That is, $$BA = BR + RA,$$
so $$BA^2 = BR^2 + 2BR.RA + RA^2,$$
hence $$BA^2 - BR^2 = 2BR.RA + RA^2 \qquad (3).$$

I7,J Adding together the equations (2) (from note 177 above) and (3) (from note 178 above) we obtain
$$\begin{aligned}(DA^2 + AB^2) - (DR^2 + RB2) &= 2BR.RA + 2\,RA^2\\ &= 2(BR + RA).RA\\ &= 2RA.AB \qquad (4).\end{aligned}$$
Now, from (1) (in note 176 above) the second member of the left side of (4) is equal to DB^2; and since 2RA = SA the right side of (4) can be written as SA. AB. Therefore (4) gives us $$DA^2 + AB^2 - DB^2 = SA.AB \qquad (5).$$

is« That is, $$\begin{aligned}SA.AB + SB.BA &= (SA + SB).AB\\ &= BA^2 \qquad (6).\end{aligned}$$

181 Eliminating SA.AB from equations (5) and (6) (notes 179 and 180 above) gives
$$DA^2 + AB^2 - DB^2 = BA^2 - SB.BA,$$
that is, $$DA^2 + SB.BA = DB^2 \qquad (7).$$

182 That is, $$SB.BA = SA^2 \qquad (8).$$
The result follows directly from the definition of proportional (golden) section.

183 Substituting the value of SB.BA from equation (8) (in note 182 above) into equation (7) (in note 181 above) we obtain
$$DB^2 = DA^2 + SA^2.$$
(continued)

Regarding the side of the star Pentagon BF: this is composed from BD, or BQ, the side of the Pentagon, and from QF, the greater part derived from it by proportional section: by [Euclid] XIII.8[iii4]*: which result can also be proved from the triangle [used in the construction] of the five-cornered figure, [triangle] FBH, as above.*

So since the square of the side of the Pentagon is equal to the square of the semidiameter, which is expressible in length, plus the square of the Greater part derived from it by proportional section, as in the diagram of the semicircle above,™'[3] *PG squared is equal to the sum of PA squared and AG squared,*™[6] *and the ratio of PA to AG will be equal to the ratio ofPG, the side of the Pentagon, to the side of its star: indeed the ratio of PA to AG will be equal to that of PG to* GX.[187] *Therefore GX is the side of the star, and its square is the sum of the square of GA, the semidiameter of the circle surrounding the ten-angled figure, and of the square of the line AX, composed of PA and AG. Thus, by what is proved there,*[18]* *GX is a Mizon, GP an Elasson. Individually they belong to the eighth degree of knowledge, and to its second level (ordine). Because, taken together the lines PG, GX make the sum of their squares Expressible, namely equal to the square PX, which is five times the square of the Expressible line GA: and the same two lines PG, GX give a Medial rectangle; on this basis PG, GX, taken together, belong to the sixth degree of knowledge which was discussed in XVIII above. Finally, because the side of the Pentagon and the side of the star are related as the larger part and the whole in the divine section; they accordingly also belong to the fourth degree of knowledge, when combined with one another: see section XXIX of this book. Moreover, it follows from these properties that just as the side of the Pentagon is an Elasson, and that of the Star a Mizon, so too the line composed of them will again be a Mizon, and the side of the Pentagon will be the smaller element of this compound line, considered as a Mizon; while the side of the star will be its greater Element; and similarly also, the difference between the two sides will be an Elasson, that is DQ or QF, by the same section XXIX of this book.*

XLIII Proposition

The surfaces of the Decagon and the Pentagon belong to more remote Degrees of knowledge, as does the side of the Icosigon and the remaining [sides] of figures of this class.

That is, the square of the side of the pentagon is equal to the sum of the squares of the sides of the hexagon and of the decagon, as Kepler required to prove. Kepler's proof of this result is considerably shorter than Euclid's, and different in substance (as may be seen by comparing their diagrams), but it is not markedly different in style.

[184] Euclid trans. Heath, vol. Ill, pp. 453-454.

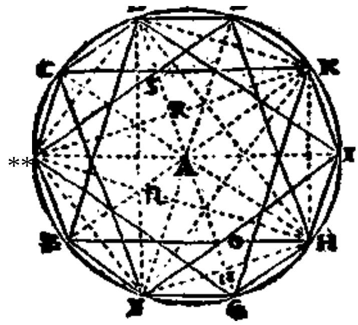
[185] In Section XXVII. The lines referred to in the remainder of Kepler's proof are all in this diagram.

186 By Pythagoras' theorem in triangle PGA.

[187] The equality of these ratios follows from the fact that PAG and PGX are similar triangles.

'»[8] That is, in Section XXVII above.

For the side of the Pentagon FH, multiplied by AN makes twice [the area of triangle] FAH, a fifth part of the Pentagonal Area. Now, FH is an Elasson, and AN is such that in square it is equal to the [square of the] Expressible line AF, less the square of the Elasson FN.[im] *Now, if the square of an Elasson is subtracted from the square of an Expressible line, the result is a new kind of line which, in square, is equal to this remainder. And the rectangle contained between a line of this new type and an Elasson will be of a still more remote kind; in fact, the area of the Pentagon will be commensurable with it, namely being in the ratio of five to two, so that it [the area of the pentagon] too will thus be of a more remote kind. Thus the side of the Decagon FG, multiplied by its perpendicular distance from the center, makes twice [the area of triangle] FAG, one tenth of the Decagon Surface, that is, [the product is] one fifth. Now, FG is a fourth Apotome; and the perpendicular to it from the center, squared, is equal to a quarter of its [i.e. FG's] square less than the square of the semidiameter. But if the square of an Apotome is subtracted from the square of an Expressible line, the line which, in square, is equal to the remainder, is of a new kind beyond those listed so far; and if such a line were to make a rectangle with an Apotome, it [i.e. the area of the rectangle] would be of a still more remote kind, and with it also five times it, that is the Area of the Decagon.*[190]



189 By Pythagoras' theorem in triangle AFN.

190 Caspar's note on this passage (KGW 6, p. 46, l. 13, note at p. 522f) indicates that he believed Kepler to be claiming that AN was a line of a new kind. However, AN is the perpendicular from the centre, A, to the side of the pentagon, FH. Kepler is concerned with the decagon, and its side FG. The perpendicular to the side is not shown in his diagram. Let us suppose its length to be h. Then it is clear that, since the perpendicular bisects FG, dividing the triangle FAG into two congruent right-angled triangles, we have, by Pythagoras' theorem in either of the triangles,

$$h^2 = AF^2 - \tfrac{1}{4}FG^2 \qquad (1).$$

This is the relationship to which Kepler has just referred. (A similar equation relates AN to the radius of the circle and the side of the pentagon.)

The area of triangle FAG may be found in two ways, either as half the product of h and FG or as half the product of AG (= AF) and FN (= $\frac{1}{2}$FH). FN can be found by Pythagoras' theorem in triangle FAN, using Caspar's value of AN, namely AN = $\frac{1}{4}\rho(1 + \sqrt{5})$, where ρ is the radius of the circle. Thus we have a second equation relating h and FG, and both lines can be found. The value of h is $\frac{1}{2}\rho(1 + \frac{1}{4}\sqrt{6 + 2\sqrt{5}})$. That is, h is the side of a square whose area is equal to that of a rectangle contained by a rational line and a binomial. Euclid considers the sides of squares equal to such areas in *Elements* X, 54 to 59 (Euclid trans. Heath, vol. III, pp. 116–129). The nature of the side depends upon the nature of the binomial used for the rectangle. If, as in the present case, the binomial is a fourth binomial (see *Elements* X, Definitions II, 4, Euclid trans. Heath, vol. III, p. 102), then the side of the square is what Kepler calls a Mizon and Heath calls a "major" (Elements X, 57, Euclid trans. Heath, vol. III, pp. 125–127). Thus Kepler is mistaken in supposing that the line concerned is of a kind that he has not previously discussed.

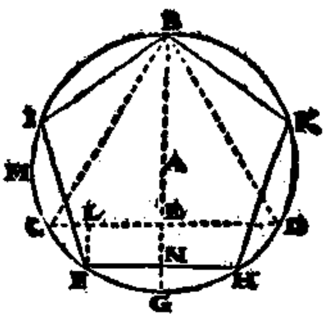
Finally, since half the side of the Decagon is a fourth Apotome and the square of the Apotome, stretched out to the length of the Diameter (which is Expressible in length), gives a width which is a first Apotome,[191] *that is to say the sagitta of the tenth part of the Circle*[192]*: indeed, the side of the Icosigon, squared, is equal to the sum of the squares of half the side of the Decagon, a fourth Apotome, and of this sagitta, a first Apotome. The surface composed from [i.e. with sides equal to]Apotomes of different kinds, and thus incommensurable with one another, will not be equal to the square of any line like those listed already; but [will be equal to the square of] some line of a completely new kind: and thus also of lower degree (ignobilior).*

How much more will this apply to the forty-sided polygon (Tessaracontagon) and the others of this class?

XLIV Proposition

The sides of the Pentekaedecagon and its stars, namely the chords subtending two, four, or seven fifteenths [of the circle],[193] do have a Geometrical description, but not apart from the circle; and in the circle, also, not through their angles, thus [the geometrical description] is not intrinsic *(impropriam)* and the knowledge is of a different kind, of a more remote degree than that of all the preceding sides. The tri-acontagon and the remaining figures of this class are of even more remote degree.

For it is described from figures before it, the relevant ones having a number of sides that is not obtainable by doubling, because 15 is an odd number, half of it not being a [whole] number[194]*: that is from the Triangle BCD and the Pentagon BIFHK, each star ting from the same point B. For if you subtract one third [of a circle], [the arc] BC,from two fifths, [the arc] BIF, that is 5 fifteenths from 6 fifteenths, the remainder is CF, 1 fifteenth. So joining the angles C, F gives the line CF as the side [of the pentekaedecagon]. Here neither the size of the angle nor the number of Angles in the figure is concerned in the process of description; nor do I construct any triangle in accordance with this number, as was done for the previous figures.*



191 Kepler means that if one constructs a square with side equal to half the side of the decagon (an apotome) and then constructs a rectangle, with the same area as this square, having one side (its length) equal to the diameter, then the other side of this rectangle (its width) will be a first apotome. The corresponding general result is proved in *Elements* X, 97 (Euclid trans. Heath, vol. Ill, pp. 212-215).

192 The *sagitta* (literally "arrow") is the part of the perpendicular bisector of the chord that is cut off between the chord and the arc of the circle. Thus, in Kepler's diagram, the line NG is the sagitta of one fifth of the circle.

193 The stars are {15/2}, {15/4}, and {15/7}.

194 Kepler uses *numerus* ("number") in the sense of the Greek ctpi6u.6c;, to mean a positive integer.

But neither can it [this figure] be described in any other way. Thus knowledge of it is also of a remote and low kind. For since FH, a side of the Pentagon, is parallel to CD, a side of the Trigon, because each figure has an odd number of sides and begins at the same point B: therefore let there be drawn from F a perpendicular to [CD, meeting it in the point] L, and from B a diameter through the center A, cutting the lines [CD, FH and the circumference] in the points E, N, G. Therefore the side CF, squared, will be equal to the sum of the squares of CL and FL[i9b]*; but CL is the magnitude by which CF, Expressible in Square, exceeds FN, that is LE, an Elasson: so CL is [a line] of a completely new kind. On the other hand, AN is a line which, squared, is equal to the remainder when a surface whose side is an Elasson is subtracted from an Expressible one: so it is of a new kind.*[196] *But EN is what remains of this new [kind of line] after subtracting AE, Expressible in length. So EN is two steps more remote. Finally CF, the side of the Pentekaedecagon, squared, is equal to the sum of the squares of CL and EN, both of new kinds; so in the former case it [CF] is twice, and in the latter case three times [more remote], and is thus [in all] five times more remote. Furthermore, the properties of different classes, those of the Trigon and of the Pentagon, are combined, so that knowledge [of the polygon] is of a different kind. What should the decision now be about the sides of the Triacontagon? Since the degree of remoteness always increases with the doubling of the number of sides of an earlier figure.*

But the chord subtending seven fifteenths, that is 14 Thirtieths, uses the side of the Triacontagon, and is posterior to it.[197] *And the chord subtending 7 Thirtieths is obtained from it by bisection [of the arc]: the same procedure generates the chord subtending 8 Thirtieths, that is 4 fifteenths, from which the chord subtending 2 fifteenths can also be obtained by bisection [of the arc]. However, this last also has another origin; for example, the chord subtended*

[195] By Pythagoras' theorem in triangle CLE

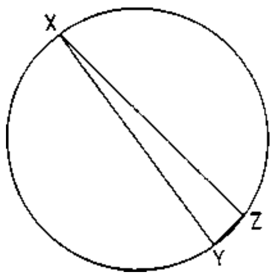
[196] In fact, AN is a binomial (see Caspar's note, not on this passage but on KGW 6, p. 46, 1.13, note at pp. 522-523).

By Pythagoras' theorem in triangle FAN we have

$$AN^2 = AF^2 - FN^2.$$

AF^2 is an expressible surface (being the square of the semidiameter). Since FN = 2FH, and FH is an elasson (as Kepler showed in section XLII above), FN^2 is a surface whose side is an elasson. The subtraction of surfaces of these particular types is not discussed in *Elements* X, though Proposition 108, which discusses subtracting a medial area from an expressible one, could be seen as suggesting a possible method of investigating the problem (see Euclid trans. Heath, vol. Ill, pp. 235-236).

[197] This can be shown by using the method described in note 153 above, making X, Y, Z vertices of the thirty-sided polygon, X and Y being opposite vertices, so that XY is a diameter of the circle, and Z being a vertex next to Y.



Since XY is a diameter, angle XZY is a right angle, and we have, by Pythagoras' theorem in triangle XYZ,

$$XY^2 = XY^2 - YZ^2.$$

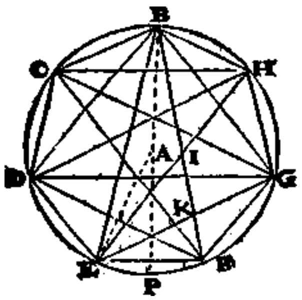
This relates the chord subtending seven fifteenths of the circle (XZ) to the side of the thirty-sided polygon (YZ), using the diameter (XY).

Place M to be the point midway between I and C.

by the arc MF, squared, is equal to the sum of the square of CF, the side of the Pentehaedecagon, and the Rectangle contained by the same CF and FI, the side of the Pentagon.^* In either way it [i.e. the chord subtending two fifteenths] is posterior to [i£. of lower degree than] the previous figures.

XLV Proposition

The Heptagon and all figures the number of whose sides are Primes (so-called), and their stars, and the complete classes [of figures] derived from them, have no Geometrical description independent of the circle: in the circle, although the quantity of the side is determinate, it is equally impossible to evaluate.[199]



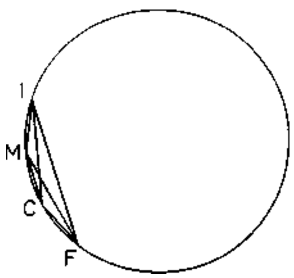
This is a matter of importance, for it is on account of this result that the Heptagon and other figures of this kind were not employed by God in ordering the structure of the World, as He did employ the knowable figures explained in our preceding sections.

So let the Heptagon be BCDEFGH, and let all angles be joined with one another, and let A be the center of the circle, and BAP a Diameter, and let A be joined to E.

First of all, such figures do not possess any non-

[198] Kepler is considering the cyclic quadrilateral IMCF (not shown fully in his figure) and using a theorem from the *Almagest,* namely that in a cyclic quadrilateral the rectangle contained by the diagonals is equal to the sum of the two rectangles contained by the pairs of opposite sides (*Almagest* I, Ch. 10, Ptolemy trans. Toomer, pp. 50-51; the theorem is sometimes called "Ptolemy's theorem").

By this theorem, we have



$$IC.MF = IM.CF + MC.IF. \qquad (1).$$

Now $IC = MF$ (2),

because M has been constructed as the mid point of the arc IC and C is a point of trisection of the arc IF.

Since M is also a point of trisection of the arc IF, we have $IM = CF$ (3),

and $MC = CF$ (4).

Substituting in (1) from (2), (3) and (4) gives

$$MF.MF = CF.CF. + CF.IF,$$

that is, $MF^2 = CF^2 + CF.IF,$

as Kepler claims.

[199] This theorem is of some historical interest, since proofs of impossibility become usual only in the nineteenth century. It is important also for the light it sheds upon the relationship Kepler considered to obtain between geometrical and algebraic methods of solving problems (see below). However, the theorem Kepler sets out to prove is, in fact, not true. Carl Friedrich Gauss (1777-1855) was to show that it is in general possible to contruct regular polygons with a prime number of sides if the number of sides takes the form $2^{2^n} + 1$. Thus one may construct figures with 3 sides ($n = 0$), 5 sides ($n = 1$), 17 sides ($n = 2$), 257 sides ($n = 3$) and so on (see CF. Gauss, *Disquisitiones arithmeticae,* 1801, paras. 337, 364, 365, reprinted in Werke, I).

intrinsic construction like that mentioned above[200]*: for the number of their sides and angles is one of the primes: but no pair of the previous figures divides the complete circle into parts that can be counted by any Prime Number: instead they [the resultant figures] correspond to a Number which is a Multiple of the Numbers [corresponding] to each figure.*

But nor do figures of this kind have a proper construction through the number of their angles: because whatever can be extracted from this is vague and non-unique and very ill-determined.

For let the Heptagon be divided up into its five triangles, two on the outside being isosceles and Obtuse-angled, namely triangles BDC and BGH, one on the inside being isosceles and Acute-angled, namely BEF, and two Scalene triangles lying in between, namely BED and BFG. So since the [arc of the] circumference on which the sides containing the angles stand, the angles themselves being on the opposite part of the circumference, takes its measure from its angle, [we may note that] the angle BEF stands on three parts [i.e. sevenths] of the circumference, BH, HG, GF; the angle BFE similarly [stands] on the three [parts] BC, CD, DE; while EBF is on one [part] EE Therefore BEF is a triangle such that each of its base angles is equal to three times the angle at its vertex. Similarly we may show that the Scalene triangle BED has angles in continuous double proportion. The simple angle is the one at B, the double at E, and the quadruple at D, being double the angle at E.

Thus if this figure [the heptagon] has a precise *(certam)* description independent of the circle, as did the pentagon above, it is required (as has already been pointed out by Campanus, Girolamo Cardano, and Foix de Candale)[201] that first of all it must be possible to construct such triangles, as a triangle was constructed for the Pentagon having each of the angles at its base equal to twice the angle at the vertex. But for that Pentagon Triangle we obtained from the angles a precise

[200] In Section XLIV, for the pentekaedecagon.

[201] In *Elements* IV, Euclid considers the construction of triangles, the square, the regular (convex) pentagon, the regular hexagon and the regular pentakaedecagon. Many of his editors and translators added further constructions of their own. The versions to which Kepler refers here were well known in his time. Campanus' translation of Euclid dated from the thirteenth century and was made from an Arabic version of the *Elements.* It was first printed in Venice in 1482 and numerous further editions appeared in the sixteenth century; Francois de Foix (1502-1594), comte de Candale, made many additions to the *Elements* in his handsome version of the work (Paris, 1566). In particular he supplied elaborate extensions of Euclid's treatment of regular polyhedra in Book XIII.

Girolamo Cardano (1501-1576) gives a brief account of the regular heptagon in Book XVI ("On the Sciences") of his *De subtilitate libri XXI* (Nuremberg, 1550; see p. 306 for the heptagon). The discussion merely states proportional relationships between the sides of the scalene "elementary" triangle and between the arcs they subtend. There is a more substantial discussion of the heptagon in Cardano's *De proportionibus* (Basel, 1570) and a further reference to the figure in *Encomium geometriae* (a lecture given in 1535 but not printed until 1562). It seems to be these latter two works that Kepler has in mind in considering the proportion between the sides of the figure (see next note).

proportion for the sides: in this Heptagon triangle, we have no precise proportion. For let I, K be the points in which BF is cut by EH, EG the trisectors of the angle BEE So in triangle FEI, because the angle FEI is bisected: so in it the ratio of FE to EI is equal to the ratio of FK to KI. But EF is equal to the whole of FI. For angle FEI is 4 sevenths of a right angle, and angle EFI is 6 sevenths, therefore EIF is also 4 sevenths. So the sides *(crura)* FE, FI opposite the equal angles are equal. For the same reason EI and IB are equal: so the ratio of FI to IB is equal to the ratio of FK to KI. Further, in triangle KEB, because angle KEB is bisected by the line EIH: therefore the ratio of KE to EB is equal to the ratio of KI to IB. But KE and FE are equal, because triangle KEF is isosceles and similar to the triangle EBF; indeed EF was equal to the line IF, and EB is equal to the line FB; so the ratio of IF to FB is equal to the ratio of KI to IB. So, for the same line BF, the chord subtending three sevenths of the circle, we have found two proportionalities, of three parts: first that the ratio of the mean line, KI, to the least one, KF, is equal to the ratio of the greatest one, IB, to the line IF, composed [i.e. the sum] of the two smaller ones, that is to the line FE, the side of the heptagon *(septanguli):* second that the ratio of the greatest line, IB, to the mean one, IK, is equal to the ratio of the whole line, BF, to the line FI, composed [i.e. the sum] of the two smallest. This kind of proportionality seems to carry the implication that there is a unique precisely determinate proportion between the lines EF and FB; and Cardano, who, when he discussed this matter concerning the sides of the Scalene triangle BED, gave it the name Reflexive Proportion, boasted, falsely, that he had found the side of the heptagon *(septanguli).*[202] For no precise quantity follows for either

Cardano's Heptagonal Reflexive proportion.

[202] There is a brief account of the heptagon in *De subtilitate* (p. 306, see previous note). Cardano defines "reflexive proportion" in his *Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum* . . . (Basel, 1570, together with the second edition of the *Ars magna).* It is the subject of the twentieth definition (p. 3) and reappears in Proposition 66 "To consider the proportion of the sides of the heptagon and the subtended [arcs] and what follows from reflexive proportion" (pp. 55-56). Cardano supplies a diagram of the regular heptagon, with a circle through its vertices. His discussion relates only to the proportions to be found in the figure. He does not mention any method of constructing it. However, in proposition 106, where Cardano discusses proportions found among angles and sides of triangles, he mentions the isosceles elementary triangles of the pentagon and the heptagon, stating that these triangles, which he implies are both constructible, allow the figures themselves to be constructed. For good measure he also implies that an enneagon can be constructed. However, no actual constructions are given.

Cardano makes some further remarks about the heptagon in the final paragraph of his *Encomium geometriae* (a discourse delivered to the Academia Platina of Milan in 1535, published as one of the short pieces following the main, non-mathematical, work in *Somniorum Synesiorum, omnisgeneris insomnia explicantes, Libri III...,* Basel, 1562, pp. 231-242, reprinted in *Opera,* vol. IV, Lyon, 1663, pp. 440-445). Here he again mentions the proportion obtaining between the sides of the scalene elementary triangle of the regular heptagon, adding that if a triangle is constructed with sides in this proportion, and a circle drawn to circumscribe it, the proportion of the complete the line EF or IF; because what we think is new information given in the second relationship is the same as the information given in the first. For, whatever 4 proportional quantities[203] are related to one another in such a way that [the sum of] the first two is equal to the third: it also holds that the ratio of the first to the third, and of the second to the fourth, is equal to the ratio of the third to the quantity composed [i.e. the sum] of the third and the fourth, which composite quantity becomes a fifth member [of the series set up by the relationships]. So the number of Cases[204] is infinite, either in terms of commensurable quantities or in terms of incommensurable ones. And in fact the number of cases for commensurable terms is the same as that of superparticular proportions, that is the same as the number of uneven square Numbers.[205]

	BF.	BI.	IK.	KF.
	9.	6.	2.	1.
	or 25.	15.	6.	4.
	or 49.	28.	12.	9.
	or 81.	45.	20.	16.
	or 121.	66.	30.	25. etc.
And the same as there are superpartient numbers	49.	35.	10.	4.
	or 64.	40.	15.	9. etc.

For the ratio of 15 to 9 is equal to the ratio of 40 to 24, the number that is the sum of 15 and 9. And the ratio of 40 to 15 is equal to the ratio of 64 (made up of 40, 15 and 9) to 24, the sum of 15 and 9.[206]

heptagon will be constructed *(erit in hoc trigono tota heptagoni ratio absoluta).* Like Cardano's other references to the heptagon, this one shows no awareness that constructing the figure might present any difficulty. The irritation Kepler betrays in his unfair accusation that Cardano is boasting in this matter may have been occasioned by the fact that in this particular passage of the *Encomium* Cardano is ascribing a cosmological significance to the heptagon, as having the same number of sides as there are planets and as embodying ratios to be found among the motions of the planets (he specifically mentions those of Mercury and the Moon) (Cardano, loc. cit., 1562, pp. 241-242; *Opera,* 1663, vol. IV, p. 445, column 2). In everything but its Platonic spirit, this suggestion is sharply at variance with Kepler's own cosmological theory.

203 Kepler means four quantities proportional two by two; see note 206 below.

204 That is, the number of possible solutions.

205 The text continues straight into the table, which is to be read line by line as if it were text.

206 In a *superpartient* ratio the antecedent contains the consequent plus several parts of it. For example, 5:3 is superbipartient since $5 = 3 + \frac{2}{3}.3$.

In a superparticular ratio the antecedent contains the consequent plus one part of it. For example, 4:3 is superparticular since $4 = 3 + \frac{1}{3}.3$.

These terms are part of the detailed classification of ratios derived from the *Arithmetica* of Nicomachus of Gerasa (second century A.D.). The work was available in several printed editions by Kepler's day, but the classification had by then become an established part of elementary arithmetic so Kepler need not have taken it directly from its Hellenistic source.

(continued)

This property is common to many proportional relationships, and it follows necessarily from the structure of the heptagon but, from only what has been given, it is not possible to construct the triangle belonging to the heptagon *(triangulum septangulare)*. The reason why in the Pentagon the proportion of the side can be precisely deter-

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The differences between Nicomachus' concept of ratio and that found in the *Elements* are discussed in Euclid trans. Heath, vol. II, p. 292.

Kepler's numerical example uses the final line of his table, in which BF = 64, BI = 40, IK = 15, and KF = 9.

In algebraic terms, the general result stated in the paragraph above Kepler's table is that if we have four quantities q_1, q_2, q_3, q_4 such that

$$\frac{q_1}{q_2} = \frac{q_3}{q_4} \qquad (1)$$

and

$$q_3 = q_1 + q_2 \qquad (2),$$

then

$$\frac{q_1}{q_3} = \frac{q_2}{q_4} = \frac{q_3}{q_3 + q_4} \qquad (3).$$

The first component of (3) is merely a rearrangement of (1), obtained by dividing both sides by q_3 and multiplying them by q_2.

Let the remaining part of (3) be expressed as

$$\frac{q_1}{q_3} = \frac{q_3}{q_3 + q_4} \qquad (4).$$

We can prove that this equation holds as follows.
Inverting both sides of (1) we obtain

$$\frac{q_2}{q_1} = \frac{q_4}{q_3}$$

If in this equation we add the denominator to the numerator on each side (thus adding unity to each side) we obtain

$$\frac{q_1 + q_2}{q_1} = \frac{q_3 + q_4}{q_3} \qquad (5).$$

Since we have

$$q_3 = q_1 + q_2 \qquad (2),$$

the left hand side of (5) will simplify to q_3 giving

$$\frac{q_3}{q_1} = \frac{q_3 + q_4}{q_3} \qquad (6).$$

Inverting both sides of this equation will give us equation (4), which is what we require.

Now, it can be shown that the "composite quantity," that is $q_3 + q_4$, which we may write as q_5, is related to q_3, q_4 and q_2 as q_3 is to q_1, q_2 and q_4. For we have, by definition

$$q_5 = q_3 + q_4 \qquad (7)$$

and from (3) we obtain

$$\frac{q_2}{q_4} = \frac{q_3}{q_5} \qquad (8).$$

These equations are analogous to equations (1) and (2), that is, they can be reduced to (1) and (2) by substituting q_3, q_1, q_2, q_4 respectively where in (7) and (8) we have q_5, q_3, q_4, q_2 respectively.

Equation (8) can be rearranged to give

$$\frac{q_5}{q_4} = \frac{q_3}{q_2} \qquad (9).$$

Since we have

$$q_3 = q_1 + q_2,$$

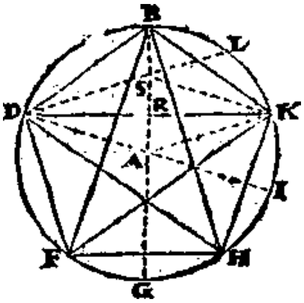
(9) becomes

$$\frac{q_5}{q_4} = \frac{q_1 + q_2}{q_2}$$

$$= 1 + \frac{q_1}{q_2} \qquad (10).$$

Since $q_1 \leqslant q_2$, this proportion between q_5 and q_4 is superpartient (see above) as Kepler claims.

mined from the angles, even independently of the circle, while the same is not true for the Heptagon and other such figures, is easily seen from what has been said already. In the triangle BFK pertaining



to the Pentagon, bisection of the angle BFK at once gives the isosceles triangles BKT and KTF, two of its [the pentagon's] elementary triangles, and it follows from the equality of their angles BFK, BKT, that the sides BK, KT, TF are equal: but in the case of the Heptagon triangle, trisection of the angle produces three elementary triangles, two isosceles triangles BEI, KEF and one scalene triangle, IEK, nor does it follow from the proportion between its [the scalene triangle's] angles that there is any particular proportion between the sides, as is known by Geometry.[207] Thus, since the angles of this figure have no significance independently of the circle; so the required triangle cannot be constructed independently of the circle. So this figure cannot be inscribed in a circle, by means of anything prior to itself in regard to knowledge or description, but this vague proportion is narrowed down into a single result only by some procedure for inscription and thus we have a circular argument; for in order to find what is required to carry through the inscription we are instructed to make use of the inscription procedure itself, as if it were already possible.[208]

As the Figure on Page 54 above

So the ratio between the Side [of the heptagon], EF, and the side of the star, FB is latent; is latent, I mean, in quantitative matters, so that by reason of the relevant principle regarding quantities, that is [the method involving the use of an] indeterminate magnitude,[209] it

[207] Kepler means that there is no proportion that is, by his definition, knowable. He must certainly have been aware of the sine rule, which states that the ratio between two sides is equal to the ratio between the sines of the angles opposite the sides; but as the sines are not, in general, knowable quantities their ratio is not knowable either.

[208] In a rigorous geometrical derivation of a magnitude, which is what Kepler is discussing here, the procedure proposed does, indeed, involve a circular argument. However, the procedure is in fact akin to the iterative methods that Kepler used in deriving (approximate) numerical solutions, from numerical data, in the *Astronomia nova.* It seems possible that the difference between Kepler's attitudes to the problems in question reflects the difference he saw between the status of geometry and that of arithmetic (see Field, 1994). Kepler seems to refer to iterative methods in his next paragraph here.

As Caspar notes (KGW 6, p. 524, referring to p. 50, 1.3), Kepler's proof is incomplete since he has not proved that it is impossible to construct the scalene triangle in question. Perhaps he thought that fact was obvious from the *Elements}* It is, however, interesting that Kepler has linked the problem of constructing a regular heptagon with the classical problem of trisecting an angle, which also proved to be insoluble by the prescribed ("geometrical") means (i.e. using straight edge and compasses).

[209] "Here, using an "indeterminate magnitude" seems to mean employing a method of successive approximation.

is in fact possible to construct the side of the heptagon in correct proportion to the diameter of the circle: since let there be given a magnitude that is certainly greater than the side of the heptagon, and one that is certainly less than it, in the same Circle: and further, subdivision proceeding to infinity can always give magnitudes greater than the side EF or less than it: but, on account of the formal properties of quantities, it is simply impossible [to find such a procedure of subdivision], because the figure of the heptagon, and similar figures, are completely lacking in any mean quantities which might lead to demonstrating or finding a proportional relation for the side of the figure [i.e. its relationship to the diameter of the circle] and thus to constructing it or demonstrating that it is knowable *(noscibilis).* Since this is so, it is not possible to inscribe a 14-sided figure in a circle with diameter AP, the side being EF, nor for two neighboring sides [of such a figure] to subtend a chord EF, which would be the side of the Heptagon inscribed in the circle: nor will it be possible for this side [i.e. of the 14-gon] to be compared with the diameter, since by its Nature its relationship to the Diameter is unknown.

So no Regular Heptagon *(Septangulum)* has ever been constructed by anyone knowingly and deliberately, and working as proposed; nor can it be constructed as proposed; but it can well be constructed fortuitously; yet it is, all the same, [logically] necessary that it cannot be known whether the figure has been constructed or no.

Algebra.

Here it might be suggested that I should use the Analytic art called Algebra after the Arab Geber, its Italian name being Cossa: for in this art the sides of all kinds of Polygon seem to be determinable. For example, for the Heptagon the following procedure is adopted by Jost Biirgi, Instrument maker *(Mechanicus)* to the Emperor and to the Landgrave of Hesse[210]; who is noted for his very ingenious and surprising achievements in this matter.

Burgi's Cossa.

In the diagram on p. 56 above.

First he assigns the value 2 to BP, the diameter of the circle, so that AB shall be a complete unit, which will be divided into parts by an indefinite [procedure of] subdivision, and these parts will give a numerical value for the length of the side BC. Then he assumes that the ratio of AB to BC is known, though this ratio is in fact what we are required to find. And he sets up the series of ratios so that the ratio of AB [taken as] 1 to BC [taken as] 11$, is equal to the ratio of *YBf* to IS, and IS to ld£, and lc£ to lSS, and lSS to 1J c^ and so on for ever, which we shall express in a more convenient notation using Roman numerals, thus: 1, lj, lij, liij, liiij, lv, lvj, lvij, and so on.[2"]

'-'"Jost Biirgi (1552-1632) was not only an outstandingly skilled maker of clocks and mechanical globes, but also a very competent mathematician. He seems, however, to have been reluctant to put his mathematical work into final form for publication. See article "Biirgi, J." in *Dictionary of Scientific Biography,* New York, 1971; J. H. Leopold (1986) and, on Biirgi's mathematical activity in particular, Balmer (1971).

-" Biirgi is using a German version of the Italian notation found in Pacioli's

Having made these assumptions we first consider the quadrilateral BEDC. So, since it has been proved by Ptolemy, Copernicus, Regiomontanus, Pitiscus, and others who have written on the theory of sines; that in any cyclic quadrilateral the single rectangle contained by the Diagonals, CE, DB, is equal to the sum of the two rectangles contained by the [pairs of] opposite sides, namely, that of DC and ED, and that of CB and DE[212]*: And again since it is known from Geometry that the sum of the squares of CO, half the chord CH, and OB, the sagitta,*[2Vi] *is equal to the square of the side CB.*[2M]

Therefore let BP be equal to 2 [units] and CB be equal to Ij, so that its square is lij, divide this by BP, it gives BO[215] *namely lij divided by 2 [units], squared [this is] liiij divided by 4, subtract this from the square of CB, lij, the remainder is 4ij - liiij [all] divided by 4, [which is] the square of CO. Now since CH is twice the line CO, the square of the line CH is 16ij — 4iiij [all] divided by 4, that is 4ij — liiij.*

Therefore, since we wish to have the square of CH or CB, that is the rectangle contained by BC and CE,[216] *multiply CB into DE, so that the rectangle contained by these lines is lij, subtract this from the rectangle contained by*

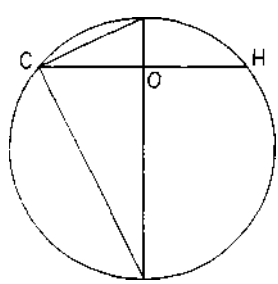
Summa de arithmetica (Venice, 1494) and elsewhere. For Bürgi ℞ = *res* (thing, Italian *cosa*—the word that gave its name to "Cossa"), ʒ = Zenzus (for Italian *censo*) = x^2, ꝯ = *cubo* = x^3, ʒʒ = *censo del censo* = x^4, ʒꝯ = x^5. The Italian notation is described in full in Bortolotti's edition of Bombelli's *Algebra* (Bologna, 1572). Kepler's notation is not only easier for the printer, who does not require special characters, but also has the advantage of making the powers of the unknown easily identifiable to the novice.

212 Kepler has already referred to this theorem, in Section XLIV, see note 198 above. In algebraic terms we have CE.DB = DC.ED + CB.DE.

213 The *sagitta* is the line segment joining the center of the chord to that of the arc.

214 By Pythagoras' theorem in triangle BOC. In algebraic terms we have $CO^2 + OB^2 = CB^2$.

215 BO = $CB^2 \div$ BP follows from the fact that the triangles BOC, BCP are similar. Our figure reproduces part of Kepler's diagram of the regular heptagon, with the addition of the line CP. H, B, and C are successive vertices of the heptagon, BP is a diameter of the circle.



Since BP is a diameter, the angle BCP is a right angle (angle in a semicircle). Moreover, BP is the perpendicular bisector of the chord CH, so angle BOC is a right angle.

Angle BCH is the angle subtended in the major segment by an arc corresponding to one side of the heptagon, BH. Therefore this angle is equal to the angle subtended by the heptagon side BC in its corresponding major segment, namely the angle BPC. (By *Elements* III, 27. Euclid trans. Heath, vol. II, pp. 58–59.)

Thus the triangles BOC, BCP are equiangular, and therefore similar. Therefore corresponding sides are in the same ratio (*Elements* VI, 4, Euclid trans. Heath, vol. II, pp. 200–202).
So we have

$$\frac{BO}{BC} = \frac{CB}{PB}.$$

Multiplying both sides by BC will give the relationship Kepler requires, namely

$$OB = CB \div BP.$$

216 CB^2 = BD.CE, by the theorem from the *Almagest* which Kepler has used before (see notes 198 and 212 above).

BD, CE, which is 4ij - liiij, there remains the rectangle contained by CD, BE, which is 3ij - liiij, divide this by lj, that is by CD, the result will be BE, 3j - liij.

Further, we turn to the Quadrilateral DBHE. And because BE is 3j - liij, the rectangle contained by BE, DE, that is the square of the line BE,[217] *will be 9ij - 6iiij + lvj: subtract the rectangle contained by BH, DE, [which is] lij, there will remain the rectangle contained between BD, EH, which is 8ij - 6iiij + lvj, divide this by EH, which is 3j - liij, the result will be BD, 8ij — 6iiij + lvj [all] divided by 3j - liij: its square [i.e.* BD^2*] will be 64iiij - 96vj + 52viij - 12x + lxij [all] divided by 9ij - 6iiij + lvj, which was [earlier found to be] 4ij - liiij: multiply this [value] by the denominator [of the previous expression] and we have*

36iiij - 33vj + Wviij - lx equals 64iiij - 96vj + 52iij - 12x + lxij

therefore also 63vj + llx equals 28iiij + 42viij + lxij[218] *therefore also 63ij + llvj equals 28 + 42iiij + lviij.*[219] *This equation gives the quantity of the side of the Heptagon.*

Or we turn, further, to BD, EG. Now the square DG, EB[220] *is 9ij - 6iiij + lvj. But the square DB, EG*[221] *is 4ij — liiij, subtract this latter from the former, the rectangle contained by DE, BG will be 5ij — 5iiij + lvj, divide this by DE, that is lj, BG will be 5j — 5iij + lv, whose square is 25ij — 50iiij + 35vj - lOviij + lx, which earlier was [found to be] 4ij - liiij. So*

49iiij + lOviij equals 21ij + 35vj + lx

Therefore also 49ij + lOvj equals 21 + 35iiij + lviij.

This equation too gives the quantity of the Heptagon side: but Biirgi turns his attention away from the complete circle and considers it only as an arc that is to be divided into 7 [equal] parts. So since the chord subtending 2 parts can be found by this algebraic procedure *(cossice),* he seeks the chord subtending 4 parts, and finds it (by the same method as above) to be the Root of 16ij - 20iiij + 8vj - lviij. He now makes use of the Diagonal in a new quadrilateral, [two of] whose sides are chords subtending three sevenths, so that the Rectangle they contain is 9ij - 6iiij + lvj, which, subtracted from the Rectangle 16ij - 20iiij + 8vj - lviij, leaves, as the rectangle of the remaining [two] sides, 7ij - 14iiij + 7vj - lviij. He makes use of this

[217] BE = DH because each is the chord subtended by three sevenths of the circle.

[218] Kepler has moved terms from one side of the equation to another in order to make all the coefficients positive. Algebraists of the time generally show a similar preference.

[2,9] Dividing throughout by the fourth power of the unknown. Gaussian hindsight seems to suggest that Kepler should have obtained an equation of the seventh degree. It is possible that he has worked round his diagram so many times that he has introduced one "side" more than once, thus obtaining an eighth degree equation with an extra double root, or with a triple one where a double one would have been expected. (On Kepler's algebra, see the paper by Field referred to in note 208 above.)

[220] DG = EB because each is the chord subtended by three sevenths of the circle.

22i rjB = EG because each is the chord subtended by three sevenths of the circle.

chord, comparing it either with the number that expresses the chord subtended by the arc that is to be divided into seven parts, or with the figure zero, if the whole circle is to be divided into seven, as here: and then either that number or the figure zero is equal to the quantities

7j - 14iij + 7v - lvij or 7 - 14ij + 7iiij - lvj.[222]

Then he deduces from the equation, which he solves mechanically,[223] not one value for the root, but two for the Pentagon, three for the Heptagon, four for the Nonagon, and so on: for one value is BC, the second BD and the third BE.[224]

In order to make it clear that this type of investigation of the sides of the figure has absolutely nothing in common with the Definitions we gave above, in our Sections I, II and III: you will note, first, that one may ask what this algebraic chord of Biirgi's signifies? It certainly signifies that if seven lines are constructed in continuous proportion, the proportion being that between the side of the heptagon and the semidiameter of the circle, and the first proportional is made equal to the side of the heptagon: then seven lines equal to the first proportional plus seven equal to the fifth will add up to the same as fourteen lines equal to the third proportional plus one line equal to the seventh.

This statement is indeed Geometrical and can be demonstrated, no less than what went before, when we showed that the surface of the Octagon was Medial, or the side of the Dodecagon was an Apotome

[222] The procedure Kepler ascribes to Biirgi is very similar to that he has just used himself, which resembles the method employed by Cardano in his discussion of the heptagon in *Deproportionibus* (see note 202 above). The fact that the figure is no longer regular does not inhibit the use of Ptolemy's theorem, which refers to any cyclic quadrilateral. There are eight equally-spaced points on Burgi's chosen arc (two of these points will coalesce if the figure becomes a regular heptagon, inscribed in a complete circle), so Euclid's theorems about equal arcs subtending equal angles are applicable as before.

Burgi's procedure is summarized in Caspar's note on this passage (KGW 6, pp. 525-526, referring to p. 52, 1.3). See also the paper by Field mentioned in note 208 above.

[223] Kepler is presumably referring to a solution by trial and error. Clockmakers used such methods in dividing their wheels—setting their compasses and then "stepping" them round the required number of times, adjusting the opening of the compasses until the result was satisfactory. It is conceivable that Biirgi used such a method to obtain solutions to his equations. However, he could have obtained the solutions (i.e. the required sides) by the same method without going to the trouble of finding the equation. In any case, Biirgi provides an interesting example of a craftsman who was in touch with some of the leading scientific minds of his day and shared some of their interests in theoretical science. (See the works referred to in note 210 above.)

[224] The equation given is for the sides of the heptagon, and the solutions will give the sides of the convex figures, {7}, and the two stars $\{\frac{7}{2}\}, \{\frac{7}{3}\}$. There will, of course, be different equations for the sides of the pentagon and of the nonagon.

Caspar's note on this passage quotes Burgi's manuscript account of his method (KGW 6, pp. 527-528, referring to p. 52, 1.7).

of some line. For there, something was being stated about the surface or line, here something is stated about the proportion between lines.

But just as it is not enough for me, for knowing and measuring a surface, to know that it is a Medial, and not enough for measuring a line to know that it is an Apotome of some line: since there are many quantities of such a type, and there is no construction [to be deduced] from this general remark, and no precise and certain quantity for the plane or line may be elicited from it, but these properties only follow from quantities previously constructed and described: so here also, it is not enough for me to know what would happen once the seven lines in continuous proportion, according to the proportion that I require, have been set up: but since I do not yet have that proportion described by geometrical means: therefore I waited for someone to explain to me how to set up that proportion first. For thus for all previous figures the procedure was [in the order]: description, inscription in a circle, determination of a precise quantity, and a precise Geometrical means by which this determination might be carried out; finally there followed the knowledge of the properties which permitted comparison of figures one with another.

To make the distinction in this matter clearer, let us look at the side of the Pentagon, whose mode of description, described above, was that, having combined two squares, one [whose side was] the semidiameter, the other [with side equal to] half of it, to make a square shape,[225] we subtracted from the side of this square half of the semidiameter; the square of the line that Remains was combined again with the square of the semidiameter, and [the result] made into a square shape, and the side of this square would be the side of the Pentagon. All this is possible and easier to do than to explain in words, as anyone knows who is used to handling compasses. For what is easier than to construct a right angle GAM, and to take on the lines enclosing it any length AM and double that length AG, and having placed one point of the compasses in M, and opened the compasses so that the other reaches to G, draw the circle GP, extend the line MA to P, then take the length GP with the compasses and transfer it into another circle[226] whose diameter is GA?

But now see what Biirgi's Cossa tells us about the side of the Pentagon. By the Method employed above we obtain the number 5j - 5iij + lv, which is not equal to any chord; that is, if five quantities are constructed in continuous proportion, the first of them being the side of the Pentagon; the proportion being that of the side of the Pentagon to the semidiameter; then five lines equal to the first pro-

225 That is, the new square has area equal to the sum of the areas of the two original ones.

226 That is, insert it as a chord into another circle.

portional plus one [equal to the] fifth will be equal to five [equal to the] third.[227]

Again, as for the heptagon, this does not tell us how to construct the continuous proportion for which this relationship will hold, nor does it express the lengths of the proportionals in terms of things already known, but it tells us, once the [system of continuous] proportion is set out, what relationship will follow. So I am instructed to represent the relationship *(affeclio)*, for it will then come about that I obtained the proportion also.[228] But how am I to represent the relationship, by what Geometrical procedure? No other means of doing it are afforded me save using the proportion I seek; there is a circular argument: and the unhappy Calculator, robbed of all Geometrical defenses, held fast in the thorny thicket of Numbers, looks in vain to his algebra *(cossa)*. This is one distinction between Algebraic *(Cossicas)* and Geometrical determinations.

Another is that all this reasoning of Biirgi's depends upon the nature *(essentia)* of a discrete quantity, namely that of numbers; and it divides the diameter into precise small parts, as many times and as far as he wishes, generally into two parts; on which number [sc. of division] the whole process depends, and it would be changed if the Diameter were given another value *(nomen)*, or a different number of parts.[229]

But Geometry does not deal with figures in this way, as we have seen above, though it does indeed designate sides Expressible in length by Numbers; but inexpressible ones it in no way attempts to capture with numbers, but states their magnitudes according to their particular kinds, so that it is clear that we are dealing not with discrete quantities but with continuous ones, that is with lines and surfaces.

Third, so far, both the side of the Figure and the side of its related star, each had a precise description;[230] in this Algebraic Analysis, the most surprising thing is that (although this may especially frighten the Geometer) there is no one way to produce what we are asked for. All the same this is not entirely without a pattern of its own, but, as I started to explain above, the number of numbers making up what is required is the same as the number of chords or Diagonals of different lengths comprised in the figure, so that in the pentagon there are two, in the heptagon three, one for the side [of the figure], the re-

227 '["he side of the pentagon is a root of the equation formed by setting Kepler's algebraic expression to zero, but in expressing this in words Kepler has again followed the usual custom of rearranging the terms so as to make all coefficients positive (see note 210 above).

228 In this context "the proportion" means the proportion between the side of the polygon and the semidiameter of the circle in which it is inscribed.

229 The diameter was designated as having the value of two units in Kepler's example above.

230 That is, it could be constructed by means of a particular geometrical procedure. ..,; .-.'.v.-. • : r •i—...'.f. - ,

mainder for chords subtending an angle [i.e. diagonals]. So that whatever is stated concerning the particular proportion of the figure[231] holds for the proportions of all lines to the diameter.[232]

Fourth, assuming that a single proportion would [suffice to] define what is required; I am not told how to bring the matter to a conclusion but only how to stalk the quarry, from a distance. For since the kinds of line, according to their [degree of] knowledge, are found among the Inexpressibles (that is, they are not numerable but reject numbers), there will accordingly be no multiplicity of numbers that can exhaust the ratio without leaving some uncertainty in it: on the other hand, this ratio, as mentioned in our second point above, takes no refuge except in numbers, but repeatedly divides the diameter in various ways into many Myriads of Myriads of parts, to make [the numerical expression for] the ratio more and more exact[233]; but this never gives a completely exact value; and, in short: this is not to know the thing itself but only something close to it, either greater or less than it; and some later calculator *(computator)* can always get closer to it [still]; but to none is it ever given to arrive at it exactly. Such indeed are all quantities which are only to be found in the properties of matter of a definite amount; and they do not have a knowable construction by which in practice they might be accessible to human knowledge.[234]

Fifth, let us concern ourselves specifically with the heptagon and following figures of this type *(genus) [sic]*, as they follow one another in order the [series of lines in] continuous proportion will grow longer as the number of sides increases: so if the one of most interest were the last one, as, for the heptagon, the seventh of the proportionals; it would, all the same, not be possible to use it to find the intermediate proportionals. For between two [lines], which are not in the proportion of two numbers of the continuous proportion, such as that one is the cube or the fifth power[235] and so on of the other, it is not

231 See note 228 above.

232 Although Kepler is unhappy about the failure of the algebraic method to give a "knowable" result, he is clearly, like all mathematicians of all times, attracted by the prospect of greater generality, which in this case is seen in the fact that the method is applicable to all regular polygons and will give all the relevant chords.

233 That is, one takes a greater number to represent the diameter in order to get more exact fractions to express the ratio. This procedure is seen in Kepler's tables of astronomical dimensions in Book V, chapters IV and IX, where the scaling factor is the number used for the mean Earth-Sun distance.

234 Kepler's point is that these quantities are not deducible from basic geometrical principles, so that they can be known Platonically (as being mathematical entities), but instead they are contingent features of pieces of matter which could have been of some other size without thereby breaching any fundamental geometrical relationship.

235 Kepler's word is *sursolidae*, which is the standard term for the fifth power. The corresponding English word, as used by Robert Recorde in *The Whetstone of Wit* (London 1557, Giii v, 1.-4), is "sursolid." Simon Stevin, however, disapproved of *sursolidus* because he believed it to be derived from *surdus* and *solidus* (see Stevin, *L'Arithmetique de Simon Stevin de Bruges*, Leiden 1585, p. 32).

possible geometrically to set up any number of intermediate magnitudes in continuous proportion but only one or three or seven or fifteen, and so on, while in the plane it is not possible to set up two or four, five, six, eight, nine, and so on[236]; since here we are considering plane figures.

Now, between the semidiameter, of magnitude 1, and the seventh proportional, of magnitude lvij, in the [system of] proportion relating to the heptagon, there are six mean proportionals, and the ratio of 1 to lvij is not that of a number to a number[237] in a continuous [system of] proportion that is equally long; that is to say, the proportion of the semidiameter to the side of the heptagon is not like that of two numbers, that is, it is not Expressible. For if it were Expressible it would fall into one of the categories *(species)* already discussed, [those] belonging to the earlier classes, and the seven angles would not be seven but [instead] three or four, which involves a contradiction. For the proportion of the sides of the first figures *(primarum figurarum)* was [deduced] from their angles.[238] Thus it would have been necessary to construct all six mean proportionals in a single step, that is [the

[236] This result can be proved as follows. Let a, b be two integers between which we are required to set up n numbers in continuous proportion. Let these n numbers be $x_1, x_2, x_3 \ldots x_n$. Since the proportion is continuous, the ratio between pairs of successive terms in the series $a, x_1, x_2, x_3 \ldots x_n, b$ is the same for each pair.

So we have
$$\frac{a}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \ldots = \frac{x_n}{b} \qquad (1).$$
From the first of these equalities we have
$$x_2 = \frac{x_1^2}{a}.$$
If we substitute this value in the second equality
$$\frac{x_1}{x_2} = \frac{x_2}{x_3}$$
we obtain
$$x_3 = \frac{x_1^3}{a^2}.$$
This procedure of substitution can clearly be repeated for successive terms, giving us
$$x_n = \frac{x_1^n}{a^{n-1}}.$$
If we now substitute this value of x_n in the equality defined by the first and last terms of (1) we obtain
$$\frac{a}{x_1} = \frac{x_1^n}{a^{n-1}b},$$
which simplifies to
$$x_1^{n+1} = a^n b \qquad (2).$$
Thus our problem of inserting n proportionals reduces to the problem of finding the $(n+1)$th root of an integer. In general, this is soluble by geometrical means (i.e. using straight edge and compasses) only if $(n+1)$ is a power of 2, that is, as Kepler puts it, if the number of proportionals is three, seven, or fifteen and so on.

Solutions are, however, possible in other cases if a and b are suitable powers of one another. For instance, if $b = a^4$, the right hand side of (2) becomes a^5, and an exact solution is possible for $n = 4$ or 9 or 14 etc.

[237] It is necessary to remember here that by "number" Kepler means a positive integer.

[238] That is, from the arrangement of their vertices on the circle.

mean proportionals] between 1 and lvij. On the other hand, if lvij were given in magnitude; then there would be five mean proportionals between 1 and lvj. Therefore, if the ratio of 1 to lvj were then to be that of a cubic number to another cubic number, then first it would be possible to construct lij and liiij in a single step, afterwards, in three steps, three mean proportionals between 1, lij, liiij and lvj.[239] However, if lv were given in magnitude, again all four intermediate magnitudes would have to be constructed in a single step; which cannot be done, unless the proportion concerned is Expressible, as above. The other [examples] are all subsumed under these.

So we conclude that these Algebraic *(Cossicas)* Analyses make no contribution to our present concerns; nor do they set up any degree of knowledge that can be compared with what we discussed earlier.

Now it is appropriate to put a word in here for Metaphysicians in connection with this algebraic treatment: let them consider if they can take anything over from it to explain its Axioms, since they say that which does not exist [a Non-entity] has no characteristics and no properties.[240] For here, indeed, we are concerning ourselves with Entities susceptible of knowledge; and we correctly maintain that the side of the Heptagon is among Non-Entities that is not susceptible of knowledge. For a formal description of it is impossible; thus neither can it be known by the human mind, since the possibility of being constructed is prior to the possibility of being known: nor can it be known by the Omniscient Mind by a simple eternal act: because by its nature it is among unknowable things. And yet this which is not a knowable entity has some properties which are susceptible of knowledge; just as if [they were] Entities with characteristics. For if there were a Heptagon inscribed in a circle, the proportion of its sides [to the semidiameter] would have such properties. Let this indication suffice.

In case it should be supposed that these comments are blasphemous. One of my friends, a very practiced mathematician, thought they could be left out. But nothing is more habitual among Theologians than to claim that things are impossible if they involve a contradiction: and that God's knowledge does not extend to such impossible things, particularly since these formal ratios of Geometrical entities are nothing else but the Essence of God; because whatever in God is eternal, that thing is one inseparable divine essence: so it would be to know Himself as in some way other than He is if He knew things that are incommunicable as being communicable. And what kind of subservient respect would it be, on account of the inexpert who will not read the book, to defraud the rest.

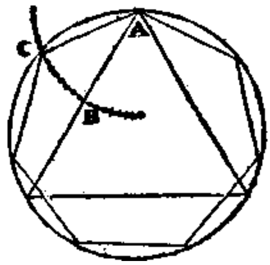
There are also other untrue propositions put forward by Geometers concerning the sides of figures like this, but which someone rela-

239 This procedure would give us the set of seven proportionals we require.

240 Kepler's citation, "Non Entis nullae dicuntur esse conditiones, nullae proprietates," seems to be a reminiscence of the dictum "A non-entity has no instances, no qualities and no action" ("Non entis nulla sunt accidentia, nullae qualitates, nulla EVEpyeia") which is cited by Clemens Timpler in his *Metaphysicae systema methodicum* as an ordinary rule among lawyers and theologians (C. Timpler, *Metaphysica systema methodicum,* Hanau, 1608, Book III, Chapter III, Problem 17).

We are grateful to Professor Charles Lohr for suggesting Timpler's work as a possible source for identifying Kepler's quotation. If Kepler did actually take the dictum from Timpler's book, he was presumably quoting from memory—and the title of Timpler's work might account for Kepler's ascribing the words to metaphysicians.

tively experienced in the Mechanical [art] would reject though because they are Mechanical they are pressed on the young[241]: as when Albrecht Diirer puts the side of the Heptagon, AC, equal to half of AB, the side of the Trigon drawn in the same circle.[242] That this is in fact considerably too short is apparent even from Mechanics[243]: however, lest anyone be misled by a rather crude practical trial; he can recognize its falsity even by this reasoning alone, without any manual procedure. From the number of its angles the side of the Trigon is proved to be Expressible in square: therefore so is half of it. The side of the Heptagon is not Expressible in square, precisely because it belongs to the Heptagon: and because seven is not six, nor five, nor three. For prime numbers give rise to sides of [particular] kinds; but these kinds [of line] are incommensurable with one another, and no one of them is the same as another.



Albrecht Durer's definition of the side of the Heptagon.

For the fallacies put forward by Carolus Marianus of Cremona and Francois de Foix, Comte de Candale, concerning the Heptagon see Christopher Clavius, *Practical Geometry* Book VIII, proposition 30, and his commentary on Euclid Book IV, proposition 16.[244]

Others.

This contest also spurred into action the Most Illustrious Lord the Marchese de Malaspina, who in 1614 was the Ambassador of the

[241] Kepler seems to mean that experienced draughtsmen recognize the constructions as merely approximate, while less experienced ones are encouraged to use them because they are convenient.

[242] See Diirer, *Underweysung der Messung mit dem Zirkel und Richtscheyt* (Nuremberg, 1525), Eiii recto. This method is to be found in many earlier authors. It may be Hellenistic in origin (see Caspar's note on this passage, KGW 6, p. 527, referring to p. 53, 1.31).

243 That is, its inadequacy is exposed by trying it in practice.

In fact, calculation shows that the method gives a very close numerical approximation to the correct side for a heptagon, so the actual construction should presumably, *pace* Kepler, give a convincing result.

Durer's claim is that for the side of the regular heptagon we may use half the side of the equilateral triangle inscribed in the same circle. Let us take the radius of this circle as our unit of length.

Then the side of the regular heptagon, which subtends ^*r* degrees at the center of the circle, will be of length 2 sin ^{1}f^2°, that is 0.867767 (to six decimal places).

The side of the equilateral triangle will be 2 sin 60°, that is V3 = 1.732051 (to six dec. pi.). Half of this is 0.866025 (to six dec. pi.). The difference between these values is only 0.001742, which is about 0.2% of the side of the heptagon. (That is to say, the error is about the same as in the agreement between theory and observation for Kepler's third law.)

Kepler is, of course, perfectly correct in saying that the method does not provide a rigorous mathematical construction for the side, and thus does not lead to "knowing" it in the sense he has defined.

[244] C. Clavius, *Geometria practica*, Rome, 1604, Book VIII, Prop. 30 (Theorem 12), pp. 407-409; and Euclid, *Elementa,* ed C. Clavius, Rome, 1574, folio 142ff.

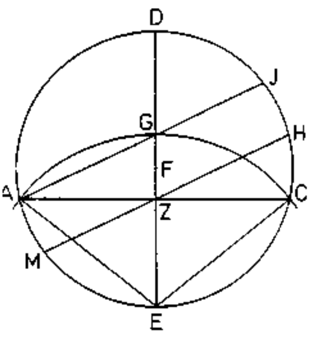
(continued)

Most Serene Duke of Parma to the Imperial court; and whose most ingenious diagram beat all the descriptions put forward by everyone else; estimating that the chord subtended by three fourteenths of the circle was equal to five quarters of the semidiameter, and thus expressible in length: so expertly was the apparatus of proof deployed that even Euclid himself might have failed to notice that something had been assumed without proof.[245]

This passage in *Geometriapractica* discusses the constructions for a regular heptagon put forward by Diirer, Marianus, and Francois de Foix. Clavius shows that all the constructions give incorrect results, but does not concern himself with the nature of the errors the constructions entail. He does not mention the work of Cardano (see note 194 above).

245 Pier Francesco Malaspina (1550-1624), Marchese degli Edificii, had a distinguished career as a diplomat. He was the duke of Parma's ambassador to both Emperor Maximilian II and Emperor Rudolf II. His funeral oration describes him as being skilled in mathematics, as was appropriate to a gentleman of his standing (see Pietro Baldelli, *Delle lodi di Pier Francesco Malaspina* . . ., Piacenza, 1624, p. 15, 1.-2).

Etiquette appears to have forbidden Kepler to be as forthright about the Marquis' proof as he had been about that of Diirer (a mere craftsman), but we may perhaps find a certain eloquence in his not giving a detailed account of it. This omission is repaired by Caspar (KGW 6, p. 528, referring to p. 56, 1.9), who draws upon material in the Pulkova manuscripts (vol. V, folios 61-62) which seems to contain Kepler's notes on his conversation with Malaspina. We have retained the letters used in Caspar's diagram.



We are given a circle center F, radius r, in which there is drawn a diameter DE. With center E. and radius f r, we draw a circular arc to cut the given circle in A, C and DE in G.

We now draw the line AG, and produce it to cut the given circle in the point J.

Malaspina claims that the arc CJ is one seventh of the circumference of the given circle. His proof of this claim is as follows.

Let us draw the line AC, to cut DE in Z. And through Z draw a line parallel to AJ, to cut the given circle in M, H. Join AE and EC.

By symmetry it is clear that DE is the bisector of angle AEC. Therefore

$$\text{angle } AEC = 2 \times \text{angle } GEC \qquad (1).$$

Now, GC is an arc of the circle whose center is E, and A also lies on the circumference of this circle, therefore

$$\text{angle } GEC = 2 \times \text{angle } GAC \qquad (2)$$

(angle subtended at center is twice angle subtended at the circumference, by *Elements* III, 20, Euclid trans. Heath, vol. II, pp. 46-47).

Combining (1) and (2) we have

$$\text{angle } AEC = 4 \times \text{angle } GAC.$$

In the circle center F, there are the angles subtended by the arcs CHJ and CDA. Therefore these arcs are in the ratio 1:4. Therefore the arcs CHJ, JDA are in the ratio 1:3.

It is here that Malaspina, as Kepler politely puts it, assumes something that stands in need of proof. What Malaspina assumes is that F is the mid point of GZ.

A measure of Kepler's politeness may be taken by examining this assumption in a little more detail. Let us grant the assumption true.

Footnote 245 (*continued*)

Now, we have constructed the circle center E so as to make EG $= \frac{5}{4}r$. Therefore GF $= \frac{1}{4}r$, so if F is the mid point of GZ then FZ $= \frac{1}{4}r$. This gives us ZE $= \frac{3}{4}r$.

Now let us consider the right-angled triangle EZC. EC $= \frac{5}{4}r$ (because C lies on the circle with center E and radius $\frac{5}{4}r$) and ZE $= \frac{3}{4}r$. Thus we must have a 3, 4, 5 triangle. So ZC $= 4 \times \frac{1}{4}r = r$, making AC $= 2r$. But r is the radius of the circle, center F, on whose circumference the points A, C both lie. Therefore AC must be a diameter of the circle (*Elements* III, 15, Euclid trans. Heath, vol. II, pp. 36–37). Therefore Z, the mid point of AC, must be the center of the circle. But its center is F. So we have an absurdity. Thus it cannot be true that F is the mid point of GZ. (In fact, ZF $<$ FG.)

It is rather difficult to believe that Kepler did not notice this contradiction. However, ZF and FG may well appear to be equal in a small diagram, a fact which might account for Malaspina's error.

Accepting the assumption that F is the mid point of GZ, it is clear, by symmetry, since MH has been constructed to be parallel to AJ, that

$$\text{arc JDA} = \text{arc MEH} \qquad (3).$$

Also, again by symmetry, we have

$$\text{arc AM} = \text{arc HJ} \qquad (4).$$

In addition, Malaspina assumes that H is the mid point of the arc JHC. Presumably this follows from the assumption that arc CH = arc AM (which would be true if FZ were perpendicular to the bisector of angle AZM). Then using (4) would give

$$\text{arc HJ} = \text{arc CH} \qquad (5).$$

Now, from the diagram it can be seen that

$$\text{arc MEH} = \text{arc MEC} + \text{arc CH}.$$

We have assumed that

$$\text{arc CH} = \text{arc AM},$$

therefore

$$\begin{aligned}\text{arc MEH} &= \text{arc MEC} + \text{arc AM}\\ &= \text{arc AEC} \qquad (6).\end{aligned}$$

Therefore, from (3) we have

$$\text{arc JDA} = \text{arc AEC} \qquad (7).$$

But we have already proved (correctly) that

$$\text{arc JDA} = 3 \times \text{arc CHJ},$$

and we know that

$$\text{whole circle} = \text{arc AEC} + \text{arc CHJ} + \text{arc JDA}.$$

Therefore we have

$$\text{whole circle} = \frac{(3 \times \text{arc CHJ}) + (\text{arc CHJ}) + (3 \times \text{arc CHJ})}{7 \times \text{arc CHJ}.}$$

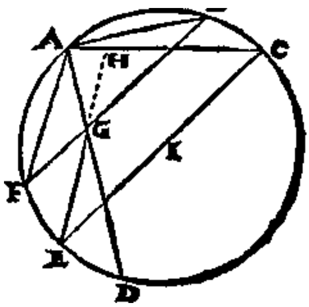
Which is the result Malaspina wanted.

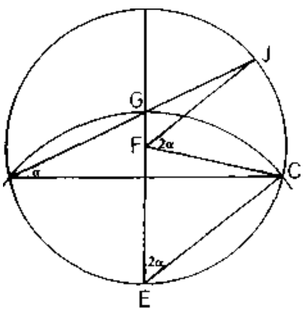
A, C, J will accordingly be considered as vertices of a regular heptagon inscribed in a circle center F. Since AC lies nearer E than D and is perpendicular to DE, it is clear that D must be another vertex of the heptagon and E must be the mid point of the arc cut off by one side of the heptagon. Thus the arc EC is $\frac{3}{14}$ths of the circle. By construction, chord EC $= \frac{5}{4}r$. Thus Malaspina's construction does, as Kepler says, make the chord or an arc of $\frac{3}{14}$ths equal to five fourths of the semidiameter of the circle. However, Kepler does not mention the value Malaspina obtains for the chord of one seventh, that is the side of the heptagon. As Caspar points out (KGW 6, p. 521), the value rather disconcertingly turns out to be exactly that obtained by Dürer's construction.

(*continued*)

Endecagon.

For the side of the Endecagon the following description is in circulation: In a circle, let there be drawn from the same point A, the side of a Tetragon AC, in one direction, the side of a Trigon AD in the opposite one, and the side of the Hexagon AB, AF in each direction: and let the angle FAB contained by the two Hexagon sides subtend another Trigon side, BF, which will cut the first Trigon side, AD, in G: let there also be drawn from the end C of the Tetragon side the diameter CE, passing through I, the center of the circle, and from the other end of the diameter, E, through the point of intersection, G, of the two Trigon sides, let there be drawn the straight line EG, cutting the Tetragon side AC in H: the line GH between these two points of intersection is said to be the side of the Endecagon. It is indeed too long, as even practical methods *(Mechanica)* show. But an expert *(sollers)* Geometer will bear in mind the kind of line that is involved, which necessarily has something in common with the sides of the Trigon and the Tetragon, though it belongs to a remote degree. But, all the same,





Our figure reproduces part of that used in Malaspina's construction, with some additional lines.

Let angle $JAC = \alpha$.

Then angle $JFC = 2\alpha$

since the angle an arc subtends at the center is twice the angle it subtends at a point on the circumference of the circle.

So, by considering the two triangles formed if a perpendicular were dropped from F to the chord JC, we have

$$JC = 2r \sin \alpha .$$

Now, in the circle center E, the chord GC subtends an angle α at a point on the circumference, A, therefore at the center, E, it subtends an angle 2α.

Triangle FEC is isosceles (because F is the center of the circle on which E and C lie) so we may find EC, as we found JC above, by considering the two triangles formed by dropping a perpendicular from F to EC.

This gives $EC = 2r \sin (90° - 2\alpha)$.

But we know $EC = \frac{5}{4}r$.

Therefore we have $\frac{5}{4}r = 2r \cos 2\alpha$,

that is $\cos 2\alpha = \frac{5}{8}$.

Using the identity $\cos 2\alpha = 1 - 2 \sin^2 \alpha$ we obtain

$$\sin \alpha = \sqrt{(\tfrac{1}{2}(1 - \tfrac{5}{8}))}$$

$$= \frac{\sqrt{3}}{4}.$$

Therefore we have $JC = \frac{1}{2}\sqrt{3}r$.

This is the value for the side of the heptagon obtained by Dürer's construction (see note 243 above), so the two constructions are mathematically equivalent.

the number 11, being a Prime, does not in any way lead one to these figures, for since it is a Prime it has nothing [sc. no factors] in common with 3 or 4. So the Geometer is confident that the description [just given for the Endecagon] is incorrect; and he may easily dispense with the labor of [checking this by] computation.[246]

It remains, therefore, that for all these objections, for all the frustrated attempts by all these scholars, the sides of figures of this kind[247] are by their very Nature unknown and unknowable. So it is not to be wondered at that what could not be found in the Archetype of the World is not expressed either in the structure of the parts of that World.

XLVI Proposition

The division of any arc of a circle into three, five, seven, and so on, equal parts, and in any ratio which is not obtainable by repeated doubling from the ones which have been shown above, cannot be carried out in a Geometrical manner which produces knowledge.

The division of an arc into two, four and eight parts, and so on, that is into a repeatedly doubled number of parts, can be carried out Geometrically, and has been used so far. It happens that not only the complete circle can be cut into three parts, by the Trigon; but also the Semicircle, as for the Hexagon; and also a quarter [of the circle], as for the Dodecagon; and also a fifth, as for the Pentekaedecagon; and also the arc of 135 degrees, as in the Octagon; and also the arc of 108 degrees, as in the Decagon. Indeed, it happens similarly that not only the complete circle can be cut into five parts, by the Pentagon; but also the semicircle, as for the Decagon; and also a third part of the circle, as for the Pentekaedecagon; and also the arc of 150 degrees, as for the Dodecagon. The same is true for the halves of these arcs, and for the quarters, and for all other parts obtainable by successive halving. But this does not come about because of a characteristic of Trisection and Quinsection, but by chance, and on account of the other properties of the figures, as already discussed.

But in the general case trisection, or division in any other proposed ratio not obtainable by repeated doubling, is impossible, as can be seen by comparison with the possibility of bisection. For that, the means used to bisect the arc, and the angle that it measures, is the straight line subtended by the arc, which [i£. the straight line] can be divided into two equal parts Geometrically: since from the equality of these two parts it follows that the parts of any arc are equal, whether it be large or small with respect to the whole circle: and from this starting point we may also deduce that in a Triangle one may argue from the equality of sides to the equality of the angles opposite them. Now, this means [i.e. the

[246] As with the other approximate solutions, Kepler is not, except by way of refutation, concerned with how close an approximation the construction may provide. He merely wishes to establish that the construction is not mathematically rigorous and therefore does not lead to "knowledge" of the side of the polygon concerned.

[247] That is, with a prime number of sides greater than five.

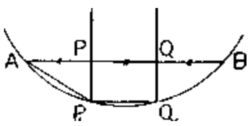
division of the subtended line] is lacking to us in other types of section. For, although a straight line, subtended by an arc, can be divided into any number of [equal] parts, and that Geometrically; yet from any proportion of the parts of the subtended line (after the proportion of equality) it is not possible to deduce a corresponding proportion of the parts of the arc; in the same way in the Triangle one may not argue from some proportion among the sides (apart from the proportion of equality alone) to the same proportion among the angles opposite them. For, if the subtended line were, say, divided into three equal parts; if the lines dividing it have been drawn perpendicular to the chord, the middle part of the arc will be smaller than the ones to either side; if the dividing lines come out from the center of the arc [i.e. the center of the circle of which the arc forms part], the middle part of the arc will be larger than the side ones. Therefore between the infinite distance[248] *and the center of the circle, there is a point such that, if two lines were drawn from it, they would divide the subtended line and its arc into three equal parts. In fact, this point is always further from the arc of the circle as the arc of the circle that is to be trisected becomes smaller, but not in constant proportion.*[249] *Thus since the arcs of the circle can be made indefinitely small (minui possunt in infinitum), the distance of this point can also increase indefinitely (excurret in infinitum): now there is no knowledge possible of something unbounded or of unbounded variation.*™ *This difficulty*

248 "The infinite distance" means the point at infinity (as it would now be called) where the parallel lines dividing the arc meet one another. Kepler had already considered such a point in his discussion of conic sections in *Ad Vitellionem paralipomena* (Frankfurt, 1604), ch. IV, section 4 (KGW 4, pp. 90-93). (See also Field, 1986.)

249 As in his discussion of conies in *Ad Vitellionem paralipomena* (on which see previous note and Davis, 1975), Kepler is using a continuity argument. When he says the change is not "in constant proportion" he presumably means that making the arc five times smaller does not make the point in question five times as far from the arc.

250 The argument that Kepler deploys here is of historical interest because it sheds some further light on his conception of points at infinity. In Ad *Vitellionem paralipomena* (1604) Kepler had introduced one such point in an *ad hoc* manner, without explicit discussion, apparently so as to provide the parabola with a second focus (see the paper by Field referred to in note 248 above).

Kepler first claims that if the chord of the arc is divided into three equal parts by lines perpendicular to the chord then the parts into which these lines divide the arc will be unequal, the central part being smaller than the other two. This may be proved as follows.



Let the given arc be AB, part of a circle with center, and let the points of trisection of the chord AB be P, Q. Let the lines through P, Q perpendicular to AB cut the arc AB in Pi, Qi.

Consider the chords APi and PiQi.

From symmetry, it is clear that PiQi is parallel to AB and is equal to PQ. That is

$$PiQi = PQ.$$

Now, by the construction of P and Q, we have,

$$PQ = AP = AB.$$

Therefore we have $$PiQi = AP.$$

Footnote 250 (*continued*)

The triangle APP_1 is right-angled, therefore its hypotenuse, being the side opposite the greatest angle, is the greatest side of the triangle. So we have

$$AP_1 > AP.$$

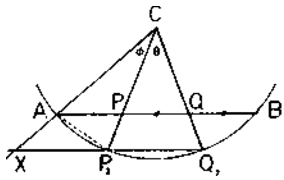
But we know that $\quad AP = P_1Q_1.$

Therefore we have $\quad AP_1 > P_1Q_1 \quad (1).$

AP_1 is the chord subtended by the arc AP_1, and P_1Q_1 is the line subtended by the arc P_1Q_1. Thus the inequality (1) gives us

$$\text{arc } AP_1 > \text{arc } P_1Q_1,$$

which is what Kepler claimed.



Now let us consider the case when lines through the points of trisection, P and Q, are drawn from C, the center of the circle to which the arc AB belongs. Let the lines CP, CQ cut the arc AB in the points P_2, Q_2. Let the radius of the circle be r; let angle PCQ be θ and angle ACP be ϕ.

Kepler claims that in this case the outer parts of the arc will be greater than the middle one, that is, $\quad \text{arc } P_2Q_2 > \text{arc } AP_2,$

which gives a corresponding inequality for the chords $P_2Q_2 > AP_2$, and for the angles $\theta > \phi$.

Draw the line CA and produce it to meet Q_2P_2 produced in X.

It is clear that, by symmetry, P_2Q_2 is parallel to AB, therefore the triangles CXP_2 and CAP are similar. So we have

$$\frac{XP_2}{AP} = \frac{CP_2}{CP} \quad (2).$$

Also, triangles CP_2Q_2 and CPQ are similar. So we have

$$\frac{CP_2}{CP} = \frac{P_2Q_2}{PQ} \quad (3).$$

Since the right hand side of (2) is the same as the left hand side of (3) the equations may be combined to give

$$\frac{XP_2}{AP} = \frac{P_2Q_2}{PQ} \quad (4).$$

Since P, Q were constructed to give $AP = PQ = \frac{1}{3}AB$, equation (4) gives us

$$XP_2 = P_2Q_2.$$

So we have $\quad$ area of triangle CXP_2 = area of triangle $CP_2Q_2 \quad (5),$

since the triangles stand on equal bases and have the same height.

But, $\quad$ triangle CXP_2 = triangle CAP_2 + triangle AXP_2,

therefore from (5) we have

$$\text{area triangle } CP_2Q_2 > \text{area triangle } CAP_2 \quad (6).$$

Now, the area of any triangle is half the product of two of its sides multiplied by the sine of the angle enclosed by those sides. Thus we have

$$\text{area triangle } CP_2Q_2 = \tfrac{1}{2}r^2 \sin\theta$$

and $\quad$ area triangle $CAP_2 = \frac{1}{2}r^2 \sin\phi.$

Therefore (6) gives us $\quad \sin\theta > \sin\phi,$

that is $\quad \theta > \phi,$

as Kepler claimed. So we have the middle arc, P_2Q_2, greater than the side ones.

Kepler then uses an argument from continuity, claiming that since lines from the center of the circle through P, Q make the middle arc too large and lines from the point at infinity (that is parallel perpendiculars to the chord) through P, Q make the middle arc too small, then there must exist a point between C and infinity such

Footnote 250 (*continued*)

that lines from it through P, Q will divide the arc AB into three equal parts. Let this point be called Z.

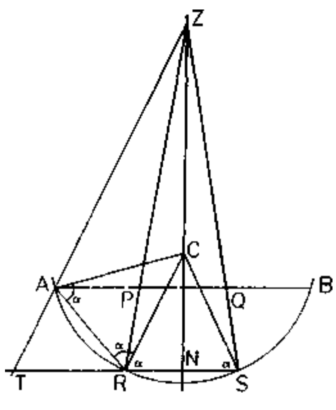
Kepler next considers what will happen to Z as the arc AB becomes smaller. He claims that its distance from the arc will increase without limit. He unaccountably regards such behavior as intolerable, though he has accepted the idea that the point where parallels meet is at "infinite" distance, that is, at a distance greater than any assignable one. Presumably the static nature of this infinite quantity was reassuring whereas the idea of unbounded variation (which nevertheless kept the point at a distance less than that of the point at infinity) encountered Aristotelian inhibitions in regard to motion of infinite bodies and comparisons between infinite magnitudes.

Kepler gives no argument in support of his contention that the distance of Z from the arc will increase without limit as the arc diminishes. In view of his not providing mathematical proofs in support of the earlier parts of his argument this may not seem sinister. Unfortunately, however, as Caspar pointed out, what Kepler claims here is untrue: as AB diminishes the distance of Z from the arc will always be less than three halves of the diameter of the circle.

The result may be proved as follows. Let the points of trisection of the arc AB be R, S. By symmetry, it is clear that Z, defined as the meet of RP and SQ, will lie on the perpendicular bisector of the chord AB, which will also be the perpendicular bisector of the chord RS, and which passes through the center of the circle, C. Let this perpendicular bisector meet RS in N. Let ZA produced meet SR produced in T, as shown. Since RS are points of trisection of the arc AB it is clear, by symmetry, that the line RS is parallel to the line AB.

Therefore triangles ZTR and ZAP are similar, so we have

$$\frac{TR}{AP} = \frac{ZR}{ZP} \qquad (7).$$



Also, triangles ZRS and ZPQ are similar, so we have

$$\frac{RS}{PQ} = \frac{ZR}{ZP} \qquad (8).$$

Combining (7) and (8) gives

$$\frac{TR}{AP} = \frac{RS}{PQ} \qquad (9).$$

Now, P and Q are points of trisection of AB, so

$$AP = PQ = \tfrac{1}{3}AB.$$

So equation (9) reduces to

$$TR = RS \qquad (10).$$

Also, since R is the point of trisection of the arc AB, we have

$$AR = RS.$$

Therefore AR = TR, and the triangle RTA is isosceles.

Since R and S are points of trisection of the arc AB, whose center is C, the triangles CAR, CRS are congruent isosceles triangles. Let their base angles be α.

Then angle ARS = 2α.

Now, angle ARS is the exterior angle at the vertex, R, of the isosceles triangle RTA. Thus each base angle of the triangle must be α, so we have

angle ATR = α.

But we have angle CRS = α,

Therefore RC is parallel to TZ.

Therefore triangles NTZ and NRC are similar.

Therefore

$$\frac{NZ}{NC} = \frac{NT}{NR} \qquad (11).$$

besets even Trisection, which is still simpler and closer to equality [i.e. bisection]. Much greater difficulty will arise in the following divisions of a general arc, say into 5, 7, 9, 11, etc. equal parts. For there can no longer be a single point from which the lines are drawn which cut the chord into the required equal parts while the same lines also cut the arc into equal parts.[251]

Now whatever techniques we can bring to bear to carry out division in the general case, techniques depending on the number that defines the division, these techniques must be general and apply equally to the lines subtended by any arc, both for a large arc which is very different from the line it subtends, and for a small arc, which differs little from the line. But leaving vague the ratio of the parts of the chord to the parts of its arc is definitely not a determination that yields knowledge. And let this be noted particularly for Trisection or quinsection etc. as carried out by Burgi's analytical method, which we discussed at length in the preceding proposition. However all the things said there apply here also; moreover, some of the things said there are more appropriate in this place, and become clearer and more significant in the division of arcs than [they were] in the division of the complete circle. For, if I pass over the

What is the character of Burgi's division of the arc into aliquot parts.

Since TR RS (by (10) above) and N is the midpoint of RS (by construction), (11) gives

$$\frac{NZ}{NC} = 3.$$

Thus $NZ = 3NC,$

and $CZ \quad NZ - NC$

$= 2NC.$

Now, however small the arc AB is made, the distance CN will always be less than the radius of the circle, and the distance CZ will thus always be less than the diameter of the circle, so the distance of Z from the arc will always be less than three halves of the diameter.

Appeals to symmetry have been introduced to shorten this proof, but it does not contain any other elements which would have been unacceptable in Kepler's time. Indeed, it may seem to follow quite simply from the short proofs of the previous two results concerning points of trisection. However, there is considerable difference between giving an adequate proof of a result that is already known (as this one was known to us from Caspar's note) and finding an adequate method of investigating the properties of a geometrical configuration in order to decide whether some result is likely to be true (as a preliminary to seeking a proof of it). Our initial investigation was essentially algebraic, involving calculations of various lengths. This attempt at foolproof caution was, of course, occasioned by our recognition that we were rushing in where Kepler's geometrical intuition had apparently let him down. For our quasi-Euclidean proof of the theorem we are indebted to Dr A.E.L. Davis.

Though this proof is fairly neat, it is not so very simple as to suggest that the theorem itself should have been obvious from an inspection of the diagram. Perhaps a very competent draughtsman drawing an accurate diagram might have noticed, if drawing several different cases in succession, that the lines we have called TZ and RC always came out parallel. Kepler, it seems, did not.

[251] The simplicity seen in the diagrams in the previous note would not obtain in corresponding diagrams for division of an arc into 5, 7, 9, 11 parts and so on. However, the dividing lines would meet two by two on the central line (the perpendicular bisector of the chord AB), so it is possible that a more elaborate version of Kepler's foregoing argument might be constructed.

points which the two cases have in common, [namely] that it is begging the question if we are told to do what it was required to find out how to do: that the properties of a continuous quantity cannot be given, in a way that produces knowledge, by discrete quantities or numbers; that whatever number is obtained for the side which determines the required part of the arc it cannot tell us more than that the side is either larger or smaller than it should be; that as [the relationship of] rough and unshaped' matter is to something which has form, and as [the relationship of] an indeterminate and indefinite quantity is to a figure, so also is [the relationship of] the analytic method to geometrical determination (the former is particularly excellent and noble in this semimechanical Cossa, but base and degraded in geometry which produces knowledge); that whereas every single chord which is less than a diameter is associated with two unequal arcs of the circle, of which one is smaller than a semicircle, and the other greater, and therefore the chord of a fractional part of the smaller one is smaller, and a part which is an equal fraction of the greater one is greater: this Analytic [method] of Burgi's tells us something general, not only about these two unequal chords but also about many other chords of a circle, which is useful for expressing [their lengths] in numbers. For example, for trisection the rule (lex) is this: If the arc be given (let it be 48 degrees) and its chord and let it be required to divide this arc into three parts, each of 16 degrees; that is if it be required to find the chord of this part, or its proportion to the whole chord of length 48 degrees: then I am required to make it so that the proportion of the chord of the whole [arc] to the chord that is to be found, that of the part [of the arc], is to be equal to the proportion of this chord to the second, and of the second to the third: now I am required to triple the chord of the part [of the arc], and from it to subtract the third proportional: the remainder is said to be equal to the whole chord. That is, from the given chord, one third is cubed, as a fraction, and the resulting number is added to the whole: the third part of this sum is a little less than the required chord. For if, instead, this [chord] itself, cubed, is added to the whole; a third part of the sum comes quite close to the correct value; and this can be repeated, indefinitely. By this procedure one comes gradually closer to the chord subtended by 16 degrees.[252] *But if you set up the number that is to be cubed as greater, and in fact to have about the value that the com-*

[252] In modern algebraic terms, Kepler is solving the equation

$$Sx - x^3 = a,$$

where a is the (given) chord of 48° and x is the (required) chord of 16°.

His method is the medieval one, derived from Islamic sources, of using successive approximations, the $(n + 1)$th approximation, x_n+i, being obtained from the rath by the formula

$$Xn + 1 = {}^{1}\text{yfl} + {}^{1}jXn^{3}.$$

The original cubic equation is, in fact, in the form which would allow a direct application of the general method of solving cubic equations first described in print by Cardano in his *Ars magna* (Basel, 1545) and probably first discovered by Scipione dal Ferro (1465-1526). However, this general method seems less attractive once one realizes that in order to obtain actual numerical solutions one would have to evaluate a number of square roots. So it is possible that Kepler's preference for the older method is not mere conservatism.

passes suggest should be a third part of the remainder of the circle when 48 degrees have been subtracted, namely 312 degrees, a third of which is 104: then in this way you will arrive at the chord of the arc of 104 degrees, and of its complement 256 degrees. Nor is this all; but if you add to 48 and 312 the complete circle, 360, you will also find the thirds of those sums, 408 and 672, namely 136 and 224, through [i.e. given as the value of] the same Term in the Cossic relationship.[253] *And in general, if one subtracts two from the number that defines the section, the number of units remaining gives the number of times one may add the complete circle to the arc it is proposed to divide, so as to discover the chords of new arcs by means of the same algebraic relationship.*[254] *From which it is clear that there is a huge difference between these algebraic terms and the degrees of knowledge which I discussed above.*

But would it not be possible to find a nobler art by which arcs may be divided into any number of parts? I reply that if all the chords of the arcs that are to be divided could all be considered in the same manner, and if we only have techniques applicable in common to all the required chords, as for dividing them in the required proportion with any number of means in continuous proportion: then no one will be able to devise anything nobler, and whoever takes any further pains about the matter is wasting his time; and in his confusion is setting up the opposite as a predicate.[255] *For from what is common [to all cases], nothing can be deduced that is applicable in any particular case.*

If, however, on the contrary we address ourselves to the specific differences among the lines subtended by the arcs which are to be divided: then the status of the question is changed, and in place of the problem of all kinds of division of an arc we substitute that of dividing the whole circle, using a Regular figure, which establishes a connection between the proposed chord and its own specific property: we have already dealt with these Regular figures above and we shall deal with them more fully below: because in this particular investigation we were seeking a means by which we may be able to draw some of those figures. So since such a means must by its nature be prior to the thing that is to be carried out by it; we should be assuming what we wish to prove if in order to procure our Means we were to seek the aid of the Regular figures.

Pappus' and Clavius' division of all kinds of arc.

Now it might here be argued against me: that Pappus of Alexandria, in Proposition XXXI of the fourth book of his Mathematical Collection, gives a way of trisecting an angle using a Hyperbola; and in Proposition XXXV a way of dividing an angle in any ratio using a Quadratrix and a Spiral: and Clavius, in Proposition 25 of Book VIII of his Practical Geometry achieves the same using the Conchoid of Nicomedes.[256]

[253] That is, the same algebraic procedure will give the chords of a series of related arcs. In modern algebraic terms the chords are all roots of the same equation.

[254] That is, in modern parlance, they correspond to the unknown in the same equation.

[255] Kepler seems to mean that such a person would be contradicting himself.

[256] Pappus, *Mathematicae collectiones,* trans, and comm. F. Commandino, Pesaro, 1588, Book IV, prop. XXXI, folio 61 *verso* to folio 62 *recto,* and book IV, prop. XXXV, folio 67 *recto* and *verso;* C. Clavius, *Geometria practica,* Rome, 1604, Book VIII, Problem 16 (Proposition 25, "To divide a given rectilinear angle into three equal parts"), pp. 399-400.

In Truth, what these authors have discovered does not establish that it is possible to carry out all kinds of division so as to obtain geometrical knowledge.[251] *To make this clear I shall first explain Pappus' mechanical procedures for trisection: then I shall draw attention to what differentiates them from constructions that give knowledge.*

Pappus' trisection of an angle.

First, Pappus himself, in the preamble before proposition 31, divides problems (which in a more general sense of the word he calls Geometrical, whereas for us the word Geometrical has a more restricted significance) into Plane, Solid and Linear: and it is stated that Trisection of an angle cannot be carried out by Plane constructions (which for me are, in the restricted sense, Geometrical, rigorous, and of the degrees I have explained), and on this he exposes the ill-conceived attempt of ancient Geometers who here labored in vain.

So he himself carries out his trisection by Solid constructions, and for section in general uses curves.

The method of trisection is this. An angle having been proposed for trisection, from a point on one of the lines enclosing the angle he draws a line perpendicular to the other such line, which perpendicular line is understood to determine the lengths of the lines enclosing the angle. And having drawn lines parallel to the shorter line and to the perpendicular, the former from the first point and the latter from the proposed angle, so that they meet and also form a right angle: now through the foot of the perpendicular he causes there to pass the surface of a Cone, a solid figure; then he inclines this applied Cone, or makes it nod, until with this same surface it defines the section known as a hyperbola in the plane, such that the two parallels just drawn are Asymptotes to it: then taking as center the point that is the foot of the perpendicular, with radius twice the length of the first line enclosing the angle, he draws in the plane an arc, to cut the line of the Conic section; and joining the center of the arc with that point of intersection, he draws a line parallel to this [i.e. the join just drawn] from the proposed angle; having done this he shows that the part cut off from the angle is one third.

Pappus makes this problem a Solid one because he used a Cone, a solid figure. But insofar as between given Asymptotes (drawn perpendicular to one another) making a right angle, through a given point lying between them, it is possible to draw the Conic section called a Hyperbola,™ in the planes even without using a Cone; the problem seems equally to be classifiable as Linear. For such a line is generated by Geometrical motion, and a continuous change in distances, that is, it is represented by a collection of points, of indeterminate number; and this is no less true [of this curve] than of the Quadratrix and the Spiral, the lines which he [Pappus] uses in Proposition 35 to carry out

257 That is, the processes of division cannot be carried out by means of rigorous geometrical methods such as those used in the *Elements* and in the present work.

[258] Apollonius' *Conies* was available in the Latin version of Commandino (Bologna, 1566) and had attracted considerable attention from mathematicians (see Field, 1987 and Field and Gray, 1987). It is clear, however, that Kepler does not expect his readers to be very familiar with the work, which he himself considered to be rather difficult (see *Astronomia nova,* Heidelberg, 1609, Introduction, folio 2 *recto,* KGW 3, p. 18, 11.8-10).

Trisection and General division [of an angle]. This is Pappus' mechanical procedure.™

So what shall we say? Surely between the given Asymptotes and through the given point only one Hyperbola can be drawn, whether this be done by adjusting the inclination of a Cone or by the infinite continuation of points? Surely there is only one point of intersection of circle and Hyperbola on one side. Surely there is only one, definite, angle made between the line that connects the points of the Hyperbola and the diameter of the figure?[260]

Indeed, I affirm that all these things are necessary and certain if the Hyperbola has actually been drawn. For earlier also, in Burgi's analytical trisection, the third part obtained on the chord bore a certain and necessary length or proportion in regard to the chord of the whole arc. But because we are not investigating what it will be, once the construction is carried out, but rather by what means, in order to give it existence, a thing not yet constructed is to be constructed: accordingly, we get nothing more from the Solid and Linear Problems of the ancients, as far as obtaining knowledge of the required line is concerned, than we got before from the Analytical method of the moderns. There is clearly only one line of a Hyperbola [that lies] between the given Asymptotes, passes through the given point, and can be drawn in their plane. But when it is not yet drawn, I am required to adjust the inclination of the Cone over the point of application until it [the hyperbola] comes into being and is drawn: alternatively, not using the Cone, I am required to change the construction lines that plot the Hyperbola by repeatedly finding points, until the curve is long enough: and the parts that lie between the points I have plotted I am required to suppose to have been plotted also: in either case, I am required to pass over by a single act or motion something which potentially involves infinite division; so that by this passage something may be attained which is concealed in that potential infinity, without the light of perfect knowledge, which the problems the ancients dubbed Plane do have.

This kind of postulate is used frequently by Francois Viete, a Frenchman, and Dutch (Belgici) Geometers of our day,[261] *in solutions of their problems,*

259 jn Hellenistic usage, which Kepler is following, a mechanical procedure, in mathematics, was one which involved the consideration of movement, such as the "nodding" adjustment of the position of the cone that Pappus employs here. Procedures of this kind had a lower intellectual status than construction methods using only straight edge and compasses (called "geometrical"). It is presumably this pejorative overtone in "mechanical" which, together with its connections with the humble tradition of practical arithmetic, led Kepler to describe Burgi's algebraic method as "semimechanical" (see Section XLV above). In fact, with the benefit of hindsight, it can be seen that from about the middle of the sixteenth century onwards algebra was definitively moving away from being merely a branch of practical arithmetic and was playing an increasingly important part in the work of learned mathematicians.

[260] A diameter of a conic is defined by Apollonius as being a line bisecting all the chords in a family of parallel chords. A diameter is called an axis if it is perpendicular to these chords.

[261] Francois Viete (1540-1603), whom Kepler is here citing for his geometrical work, is now best remembered for his important contributions to the development of algebra. Caspar suggests (KGW 6, p. 530, referring to p. 60, 1.40) that the "Dutch"

which by their very nature are not soluble except in a way that goes against the rules of the art,[262] *such as numerically or by Geometrical motions whose changes need to be guided by some kind of infinity.*

Now, in a case where everything is available that was considered conducive to achieving certainty we shall arrive at a result which is either a little greater or a little smaller than the required value, and always [becomes] closer to it; as we also said before in relation to the Analytic method of trisection.

There is no rigorous way of going from plane to solid problems.

That what I say about this solid problem of Trisection is true is, as it were, suggested by the very word "solid." For if a proportion between solids is not given in a form such as [a ratio between] two cubic numbers: we cannot, as an intellectual procedure, measure the proposed solid in terms of the other one: because two intermediate proportionals cannot be constructed exactly in the plane: though they may be present in the cubes, yet there is no passage from the plane figures to form any of those cubes without the two means: [it is] as if the bridge were broken.[2^]

There is no rigorous procedure for finding two mean proportionals.

And for finding two mean proportionals some give instructions to use Geometrical motion, thereby ordering one to do something that is useless for achieving certainty through an appropriate Geometrical act: indeed Pappus himself gives instructions that use Conic sections, to be produced with the help of two [mean] proportionals, although the Cone itself is a solid. So we are always assuming what it is required to prove; and the bridge lies on the other bank.

XLVII Proposition

Figures with an odd number of sides greater than 5 (except the Pentekaedecagon), and the chords subtended by any number of their sides, and the whole Classes associated with them,[264] come into the same category as the Heptagon and other figures with a Prime number of sides.

For if the number of sides is odd and not a Prime: it is either the smallest multiple of two odd Primes; or the square of some Prime: or it is a multiple of one Prime and the square of another, or a multiple of squares with themselves or in combination.

So if these figures were describable and inscribable, and knowable; then they would also have a proper construction from their angles, or an improper one by taking into account figures of which they form part. But they do not

geometers may beLudolph vanCeulen (1540-1610), Adriaen van Roomen (1561-1615), and Willibrord Snel (1581-1626). Kepler knew Snel personally. Both Viete and Snel wrote treatises that used Apollonius' methods and were proposed as reconstructions of lost treatises by Apollonius (Viete, *Apollonius Gallus . . .*, Paris, 1600; Snel, *Apollonius Batavus,* Leiden, 1608; both works are reprinted in the first volume of Pierre Herigone, *Cursus mathematicus,* Paris, 1633).

[262] Kepler's word is *inartificialiter,* which means "uncraftsmanlike-ly."

[263] The problem of finding n proportional means is equivalent to finding an $(n+1)$th root (see note 236 above).

[264] The class is the set of figures generated by repeated doubling of the number of sides.

have a proper construction; because they have not got a Prime number of sides, from which a construction might take shape: they do not have an improper construction, neither for those in the first group,[265] such as the 21-angle, because the figures which contribute to them either both or either of them as here the Heptagon (after the Trigon and the Pentagon, which give the Pentekaedecagon), have no proper construction [of their own], by XLV above; nor [is there an improper construction] for those in the second group,[266] such as the Nonangle [Enneagon], because there is no way of dividing a fractional arc, for example a third, into the same number of equal parts as the whole circle has been divided into, by XLV I above: nor [is there an improper construction] for those in the third group, or the fourth, because the earlier figures which form part of them are indemonstrable.

Concerning the Enneagon, whose number of sides, 9, is the square of the first odd Prime, namely three, there has been a contest among Geometers, with many exerting themselves to construct the side of this figure also. All of them did so in vain, nor would they ever have attacked this problem if they had paid attention to the difference between things that are knowable and those that are not. The side of the Nonangle [Enneagon] is not knowable.

Campanus wished to construct the Nonangle [enneagon] by Trisection of an angle[261] which is shown to be unknowable in XLVI above. Though it is trisected of necessity by the method of Pappus and Clavius, namely by using Geometrical motion, yet what has this to do with plane figures, with which we are dealing here, when solid figures are needed to make the lines which are aids to trisection, the Hyperbola, the Quadratrix, the Spiral and the Conchoid"? Indeed Campanus himself when attempting trisection does not tell us he is taking the third of an angle as if he were actually sure this was the case, which was what was really required. The place where he says this is, in my copy, towards the end of the Works of Euclid, folio 586 referring to the end of Book IV. Campanus on the Nonangle.

Giordano Bruno ofNola,[26s] in a hexagon ABCDEF, draws perpendiculars

265 That is, the numbers of their sides are the product of two odd primes. (The only *even* prime is 2.)

266 That is, figures with a number of sides that is the square of a prime.

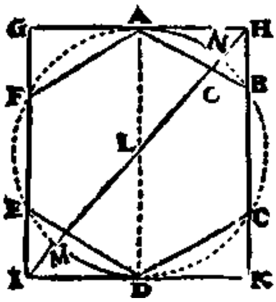
267 Campanus' version of the *Elements* went through many printed editions but it seems that there is no necessity for Kepler to be referring to any of them. Campanus' comments and additions (as well as those of many other scholars) were incorporated in the Latin edition of the *Elements* published in Basel in 1537. The page reference later in Section XLVII shows that Kepler used this edition. For the result discussed here, see Book IV, prop. 16, p. 101.

268 Giordano Bruno (1548-1600) was known chiefly as a natural philosopher rather than as a mathematician. In the paragraph that follows Kepler seems to be referring to Bruno's *De monade, numero etflgura* (Frankfurt, 1591), Chapter X (pp. 123-124), which consists of twenty-five lines of verse describing a diagram that in principle resembles the one Kepler supplies here. (In fact, the lettering on the diagrams is the same, though in mirror image.) Bruno refers to points in his diagram by names beginning with the appropriate letter: Apollo, Bonitas, Charis, Diana, and so on. His mathematical discussion accordingly consists of statements such as "Apollo is joined with Diana"—statements which seem designed to suggest non-mathematical interpretations. Bruno offers no justification for his construction. The brief description of his figure is followed by several pages of more or less cabbalistic speculations con-

(continued)

Giordano Bruno's Nonangle.

GH, IK, tangent to the circle, to the opposite sides BC, EF [each] produced in both directions; so, having drawn IH the diagonal of the parallelogram that has been constructed, he thought that the circle was cut [by this diagonal] in such a way that between A and D, the points of contact and M, N the points of intersection [of the diagonal with the circle] there are Ninths of the circle AN, DM. However, it can be shown from the diagram that since the square of the semi-diagonal [of the parallelogram], LH, is expressible, namely 7 sixteenths of the square of the diameter [of the circle] (for [angle] ABH is 60 [degrees] therefore BH is halfAB, or AL: and its square is therefore four times that of AL; so AH squared is three quarters the square of AL. But LH squared is equal to [the sum of] the squares of LA and AH),[269] *so the sine of 40 degrees, that is half the chord subtended by two Ninths of the circle, would have to be Expressible in square, namely the root of three twenty-eighths of the square of the diameter. For having dropped a perpendicular, AO,from A to LH, the ratio of the square ofLH, 7 sixteenths, to the square*



cerning nine and things that come in nines. (On this work, see Yates, 1964, esp. pp. 321–322.)

Some of Kepler's attempts to investigate the mathematics of Bruno's construction of the enneagon are preserved in the Pulkova MSS (for references see Caspar's note at KGW 6, p. 530, referring to p. 62, 1.12).

269 Kepler's mathematical arguments may be set out more fully as follows.

The interior angle of a hexagon is 120° – as can be calculated from the formula for the angle of a regular polygon (see Section XXXII and note above) – so an exterior angle, such as angle ABH will be 60°. The triangle ABH has a right angle at H, therefore its remaining angle, at A, will be 30°. So the triangle is half an equilateral triangle and we have

$$BH = \tfrac{1}{2}AB.$$

But we know that

$$AB = AL \qquad (1),$$

since the side of a regular hexagon inscribed in a circle is equal to the radius of the circle (see Section XXXVIII above).

So we have

$$BH = \tfrac{1}{2}AL.$$

Therefore

$$BH^2 = \tfrac{1}{4}AL^2 \qquad (2).$$

By Pythagoras' theorem in triangle ABH we have

$$AH^2 = AB^2 + BH^2.$$

Substituting for AB and BH^2 from (1) and (2) we obtain

$$AH^2 = AL^2 - \tfrac{1}{4}AL^2,$$

that is

$$AH^2 = \tfrac{3}{4}AL^2 \qquad (3).$$

Now, by Pythagoras' theorem in triangle LAH we have

$$LH^2 = AL^2 + AH^2.$$

Therefore, substituting for AH^2 from (3), we obtain

$$LH^2 = AL^2 + \tfrac{3}{4}AL^2 = \tfrac{7}{4}AL^2.$$

AL is the radius of the circle, so the last equation gives us Kepler's value for the square of LH (the semidiagonal of the parallelogram GHKJ), namely seven sixteenths of the square of the diameter of the circle.

of HA, 3 sixteenths: will be equal to the ratio of the square of LA, 4 sixteenths, to the square of AO, 12 sevenths and one sixteenth, that is, three twenty-eighths.[270] *So this chord subtended by the angle of the enneagon would be nobler than some of the preceding [chords], even though it is associated with them: however, since there is an odd number of sides, namely the number which is the square of the Prime number 3, there is nothing that is associated with the Tetragon and the Trigon, by bisection of arcs, though this degree of knowledge belongs to these figures.*

XLVIII Corollary

It follows therefore that the Concept, Knowledge, Determination, Description, and Construction of a figure serve to set up boundaries between the primary Orders to which the figures belong: so that the Classes of knowable figures are no more than four: three of figures that have proper demonstrations, among which are to be included the heads of families,[271] in the first [class] is the Tetragon, following the diameter of the circle, whose characteristic number is 2; in the second [class] is the Trigon, whose characteristic number is 3; in the third [class], the Pentagon, characterized by 5: and one [class] of figures with improper demonstrations, whose characteristic number is the product of two factors, 3 and 5 namely 15: for the first figure in this class is the Pentekaedecagon.

XLIX Proposition

Now, since bisection (the proper form of which is used in the first class [of figures]) is common both to the second and to the third classes: it is clear that the first class exists according to a rule which differs

[270] Let the diameter of the circle be d. Then Kepler has already shown that

$$LH^2 = \tfrac{7}{16}d^2 \qquad (4).$$

(see previous note).

Now, Bruno is claiming that his construction has made the arc AN one ninth of the circle, so the angle ALN must be $\frac{360}{9}°$ i.e. 40°.

From A we drop a perpendicular to LH, to meet it in O. By symmetry it is clear that, if AO were produced to cut the circle again, the arc between A and this point of intersection would be two ninths of the circle. Thus AO is half the chord subtended by two ninths of the circle.

Now, the area of triangle ALN is expressible either as $\frac{1}{2}.AL.AH$ or as $\frac{1}{2}.LH.AO$. Therefore these two expressions must be equal, giving us

$$AL.AH = LH.AO.$$

Squaring, we have $$AL^2.AH^2 = LH^2.AO^2 \qquad (5).$$

Now, substituting for AH^2 from (3) (see previous note), for LH^2 from (4) and writing $\frac{1}{2}d$ for AL, (5) becomes

$$\tfrac{1}{4}d^2.\tfrac{3}{4}.\tfrac{1}{4}.d^2 = \tfrac{7}{16}d^2.AO^2.$$

Therefore we have $$AO^2 = \tfrac{3}{28}d^2,$$ as Kepler says.

[271] The head of a family is a figure which gives rise to others that share its rank.

ent in the sides of the Pentagon and of its star; but it does not occur in the Decagon and its star, except if the side of the Hexagon is an intermediary; it does not occur at all in the Octagon.

Beyond these properties of the sides, there is another indicator of nobility, because the figures are differentiated by the aptness and perfection of the areas they enclose. Here, after the diameter (whose area is zero, and which, as Ptolemy notes, only divides the area of the circle into two equal parts, as it also does the circumference) the highest place is given to the Tetragon and the Dodecagon, which have Expressible areas, and the Tetragon indeed is of the greatest distinction; because its area is the same as the square of its side, for the class of area to which it belongs is to be square: so it encloses half the square of the diameter: thus the Dodecagon stands lower than the first, enclosing three quarters of the square of the diameter. In the next place there follow the Trigon, Hexagon and Octagon, whose area is Medial[275] in species, while the concepts [of the characters] of the areas of the Pentagon and Decagon have no names.

End of Book I.

"Medial" translates Kepler's "meson." See note 18 above.

BOOK II

OF THE HARMONY OF THE WORLD

by

JOHANNES KEPLER

ON THE CONGRUENCE OF HARMONIC FIGURES

Introduction

Thus far I have discussed the essential nature of individual regular figures as they are "conceived in the mind." What follows will concern a property they show when they are combined with one another, as it were their Effect in the realm of Geometry, which is Congruence[1] or Unsociability. Constructibility[2] and Congruence do not produce classes of the same width, since the former property concerns individual figures, and by the repeated doubling of the number of sides of the polygons concerned the property extends to an infinite number of figures, whereas the rules that set limits to the latter property bring many figures together into one group, but an increase in the angle of the polygon is a hindrance, and the group closes quickly. Whichever degree of knowledge and construction we choose, and there is a great difference in nobility between the ones we have discussed and those we have dismissed without giving them a name, yet rank in constructibility does not always go side by side with rank in Congruence, for the one is not the cause of the other but, rather, they have a common cause (that is, the individual character of the angle of the figure), although each depends on it according to its own rules. The necessity of this part of our speculation is clear from the over-all design of the work. For, since we have taken it upon ourselves to explain the origin of Harmony and its most powerful effects in the World as a whole, how could we omit to mention congruence of the figures which are the well-springs of Harmonic proportions? Since the Latin words congruere and congruentia mean the same as the Greek dpixoxxeiv and dpnovia? Since the effect these figures have in the realm of Geometry, and in that part of Architectonics which deals with Archetypes, is as an image of and a prelude to their effects beyond Geometry, beyond things conceived in the mind, namely their effects in things natural and celestial? Since this property of Congruence, which shows itself in structure and bodily form, is such that it, as it were, of itself encourages the speculative mind to make something external to itself, to create, to fashion a solid body. Thus it has from eternity lain hidden in the supremely blessed divine mind, as one of the Ideas, and so far

[1] *Congruentia:* the term is defined and explained below.

[2] Kepler's term is *demonstrabilitas,* but he is in fact referring to *descriptio,* which he defined in Book I, section V, where we translated the corresponding verb as "to describe." The words may have seemed more nearly interchangeable to Kepler's contemporaries than they do to a twentieth-century translator. In any case, this is not the only example of Kepler's failing to employ technical terms in an entirely consistent way (see note 79 on Book I above).

partook of the highest goodness that it might not be contained within its own abstraction but must break forth into the work of Creation, causing[3] God the Creator to enclose bodies within particular figures. So I shall give a brief discussion of the Congruence of figures, since the demonstrations are not at all difficult and require little more than a diagram of the figures.

[3] The verb used is *efficere.*

ON THE CONGRUENCE OF REGULAR FIGURES.

I Definition

Congruence takes one form in the plane and another in space. In the plane there is congruence when individual angles[4] of several figures come together at a point in such a way that they leave no gap.

II Definition

Congruence is said to be perfect when the angles of the figure which come together do so in the same way at each meeting-point, so that these meeting-points are similar to one another and the pattern of meeting-points can be continued indefinitely.

III Definition

Congruence is said to be most perfect when, in addition, the figures which come together in the plane are all of the same kind.

IV Definition

Congruence is said to be imperfect when some larger figure is surrounded by similar meeting-points but the congruence cannot be continued indefinitely or can be so continued only by introducing meeting points of different kinds. The congruence is imperfect, and of lower degree, when the larger figure cannot be surrounded in such a way that similar meeting-points are formed at all its angles.

V Definition

There is congruence in space, and a solid figure, when the individual angles of several plane figures make up a solid angle, and regular or semi-regular[5] figures are fitted together so as to leave no gap between

[4] The ambiguous use of the term "angle," which we noted in Book I (see note 8 on Book I above), continues in Book II. See also note 6 below.

[5] A "semiregular" figure has four equal sides but unequal angles (see Book I, Section III above). See also Section XXVII below.

the sides of the figures, which join up on the opposite side of the solid figure, or, if a gap is left, it is such that it can be filled by a figure of one of the kinds already employed, or, at least, by a regular figure.

The geometry of bees in their six-cornered cells with rhombic ends.

Note that there is another form of congruence, not of plane figures to form a solid figure but of these solid figures among themselves, to fill space all round a point. There are only two figures which will form such congruences: the cube and the rhombic dodecahedron. Eight cube angles[6] *will meet at a point and fill space all round it. The rhombic figure has two types of angle: eight obtuse trilinear angles and six acute quadrilinear angles. Four obtuse angles will join to fill space, and so will six acute ones. The result is like the way in which bees construct their honeycomb: the cells are contiguous and the end of each is surrounded by three opposing ends while its sides are surrounded by the sides of six more cells. Three more cells could be added at the other end to complete the figure, except that the entrances to the cells must remain open. We are not concerned here with this congruence of solid figures.*[1]

VI Definition

Congruence in space is said to be most perfect, as is also the solid formed, when, in addition, the plane figures which form the congruence are all the same shape.

VII Definition

On the one hand, the solid formed is completely regular when the plane figures are regular. All its angles then lie on the same spherical surface and are all similar to one another.

VIII Definition

On the other hand, the solid formed is semiregular when the plane figures are semiregular (see Book I, Definition III). Its solid angles are then not all the same, but differ in the number of lines they contain, though the angles are not of more than two kinds, and neither are they distributed on more than two spherical surfaces, which are concentric. The number of angles of each kind must be the same as the number of angles of one of the regular solid figures.[8]

There is no reason why we should not call this congruence most perfect,

[6] Kepler usually refers to solid angles merely as "angles," with the additional ambiguity (referred to in note 8, Book I above, for plane angles) that he does not usually distinguish between the angle proper and its vertex.

[7] Kepler had dealt with congruences of solid figures in *De nive sexangula* (Prague, 1611; reprinted in KGW 4 (pp. 259-280).

The rhombic figures are discussed, and illustrated, in Section XXVIII below.

[8] See Section XXVII below.

for its imperfection is in the faces and is not a consequence of its being solid but, rather, an accidental feature. So this semiregular congruence will equally be called most perfect.

IX Definition

A congruence is perfect, but of a lower degree, when the plane figures are regular and all the angles lie on the same spherical surface and are similar to one another, but the faces are of various kinds, though the number of each kind must be the same as the number of faces of one of the most perfect figures, that is, not less than four, which is the minimum number of planes to bound a solid figure.

X Definition

There is an imperfect congruence or figure when other conditions remain the same but the larger plane figure does not occur more than once or twice.

The solid figure formed will either be more like a part of a figure than a whole one[9] *or it will be more like a plane figure than a solid, since any solid figure is bounded by at least four surfaces. Such figures are shown in the plate, marked A and B, where the larger figure is a heptagon.*[10] *These two classes extend indefinitely as the number of sides of the larger figure increases. They each start with the trigon, which in class A gives us one of the most perfect regular congruences.*[11] *Proceeding to the tetragon we then obtain one of the most perfect congruences in class B.*[12] *All other congruences of these types are imperfect.*

A, B following page 104.

XI Definition

A congruence is semisolid when it does not satisfy all the conditions of Definition V, so that as the plane figures are fitted together the congruence does not completely join up with itself but leaves a gap. Apart from this, Definitions VI and VII apply to such congruences.

XII Definition

Plane figures are congruent when they either enclose a solid figure or fill the plane without leaving a gap, the figures themselves being regular or semiregular.

[9] If the larger figure occurs only once, the figure will be a pyramid. Kepler presumably regards this as "more like a part of a figure" because it is less symmetrical than, say, a prism.

[10] See our page 104.

[11] The figures in Kepler's class A are antiprisms. The triangular antiprism is an octahedron.

[12] The figures in Kepler's class B are prisms. The square prism is the cube.

XIII Definition

Let us call incongruent those regular plane figures inscribed in a circle (if they can be inscribed) which not only cannot form, either alone or with other plane figures of their own or another class, a solid figure, other than a somewhat imperfect one, which can be inscribed in a spherical surface, but also cannot cover the plane, either by themselves or with stars of their own class, or with figures and stars of another class around them.

Note that I have excluded the heptagon and suchlike figures, despite the fact that two parallel heptagons together with seven square or fourteen regular triangles do form a completely closed solid figure, because only two heptagons are involved, and the figure formed is discus-shaped, like a plane, not globe-
A,B. *shaped, like a sphere. See the figures marked A and B in the engraved plate following page 53.*[Vi] *The fifteen-sided figure is also excluded in the same way, despite the fact that some of its angles may be surrounded by related figures to cover the plane, because in this case the figure is not completely surrounded at all its angles.*

XIV Proposition

At least three plane angles are required to form a congruence in the plane.

For around any meeting-point the sum of the angles is four right angles. But no figure has an angle greater than two right angles, therefore two such angles are less than four right angles. So two of them cannot fill the plane, by Definition I.

XV Proposition

At least three plane angles must fit together or rise up to form a solid angle.

For two plane angles would meet not only at their sides but with their whole surfaces, which is contrary to Euclid's definition of a solid angle.[H]

XVI Proposition

The sum of angles congruent in the plane is always four right angles, never more. The sum of angles which form a solid congruence is less than four right angles.

For in a plane there are no more than four right angles around a point, therefore when the sum of the angles is equal to four right angles no gap is left, and by Definition I there is then congruence in the plane. If the angles cover the plane they do not rise from it to form a solid angle. And, on the other hand, if the angles fitted together in the plane leave a gap, that is if they come

[13] See our page 104.
[14] *Elements* XI, Definition 11 (Euclid trans. Heath, vol. Ill, p. 261).

*to less than four right angles, then drawing together the two sides round the
gap, and so eliminating it, necessitates raising the angle and making it a solid one.
Figure H in the engraved plate, following page 53,*[15] *shows three pentagons* H.
lying in the plane and leaving a gap.

XVII Proposition

A figure with an odd number of sides, around which figures of two kinds are fitted, cannot form a congruence which is the same at every angle, either in the plane or in space.

*For one angle of the figure will have the same figure on both sides of it,
which is not the case for the other angles. The reason for this can be seen in* c.
figure C of the engraved plate below.

XVIII Proposition

There are only three ways in which the plane can be filled most perfectly around a point, in each case using figures of only one kind: by using six trigons, or four tetragons, or three hexagons.

*For by XXXIII of the first book of this work the angle of a trigon is two
thirds of a right'angle, therefore the six angles of six trigons are twelve thirds,
that is four whole right angles. See D.*[i3] D.

*Similarly, the angle of a tetragon is one right angle, therefore the four angles
of four tetragons make four right angles. See E. Similarly, the angle of a hexagon* E.
*is eight sixths of a right angle, therefore three angles of three figures make twenty-
four sixths, that is four right angles. See F But the angle of a pentagon is less* F.
*than that of a hexagon, therefore three of them are less than four right angles
and leave a gap. The angle of a pentagon is larger than that of a tetragon,
therefore four pentagon angles are more than four right angles, therefore they cannot
be contained around a point in a plane, by XVI of this book. For this see H,* H.
*where the fourth pentagon is shown dotted. Similarly, the angles of a heptagon
and of all larger figures are greater than that of a hexagon, so three heptagon
angles are more than four right angles. See I, where two of the heptagons partly* i.
overlap in the plane.

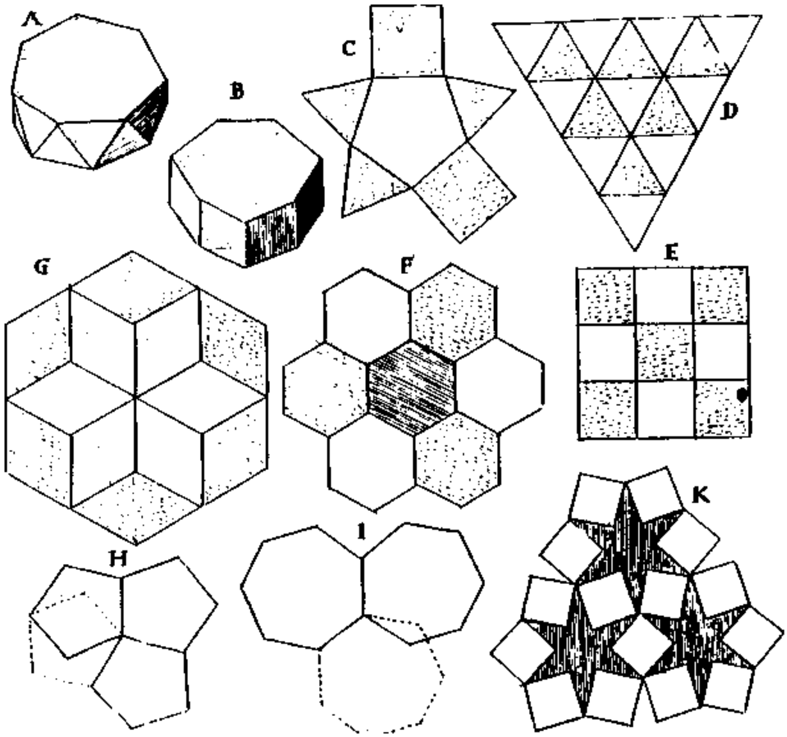
*Here we must consider rhombi made up of two regular trigons. They form
a most perfect congruence, like regular hexagons, although they are semiregular
figures. This congruence can be seen in the engraved plate, labelled G.* G.

*Here we must also consider the six-cornered stars we obtain by removing
six points from a star dodecagon.*[11] *See letter K. For where we have removed* K.
*a point we have a re-entrant angle, equal to a right angle. Therefore three tet-
ragons and three points of these stars fill the plane. For the hexagon can be
divided up into one such star and six half-tetragons.*

[15] See our page 104.

[16] See our page 104.

[17] Complete star dodecagons are shown in the pattern marked T in the Figure on our page 104.


A
B
C
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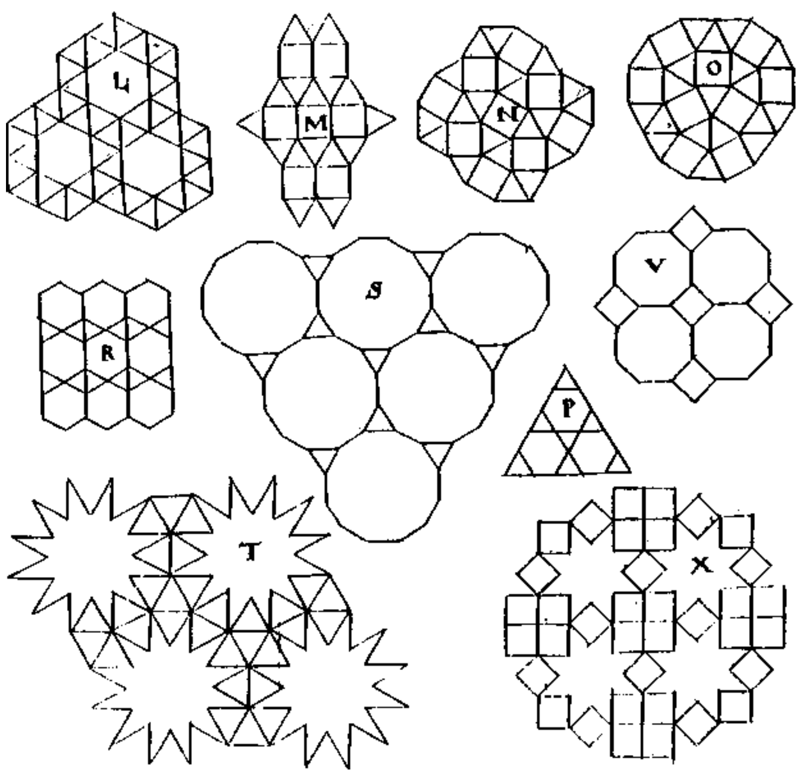
I
K


L
M
N
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R
S
V
P
T
X

XIX Proposition

There are six ways in which the plane can be filled around a point by figures of two kinds: in two ways using five angles, in one way using four angles and in three ways using three angles.

Six plane figures cannot fit together, since the angle of one of them must be larger than the angle of the trigon. The angle of the trigon, the first of the polygons, is two thirds of a right angle, so taking it six times gives twelve thirds, or four right angles. So if one of the six angles were larger, that is if it were the angle of a higher polygon, the sum would be more than four right angles, so the plane is not covered, by XVI of this book.

1. Five figures fit together if four trigon angles are combined with another angle equal to two trigon angles, that is a hexagon angle. The form is shown in L. **L.**

2. Again, five figures fit if three trigon angles are combined with two tetragon angles, because the last two add up to three more trigon angles. The form is as shown in M, or N, both of which forms can be extended uniformly, or it is **M,N.** *as shown in O, a form which cannot be extended uniformly.*[18] *But if you take* **O.** *two trigon angles with three tetragon angles they will come to more than four right angles. The sum is even larger if you add two*[19] *larger angles to two trigon angles.*

3. Four figures of two kinds fit together if two trigon angles are combined with two hexagon angles. The form is either as shown in P or as shown in R. **P,R.**

Whatever other four angles you fit together you always get more or less than four right angles so you do not fill the plane.

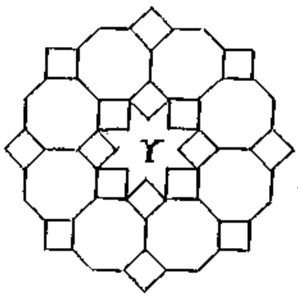
If we join up three angles, taking care not to use more than two kinds of angle, we may begin by ruling out cases which use two trigon angles or two tetragon angles, for these do not come to more than two right angles and the gap they leave for the third angle is therefore too large for any one angle to fill on its own.

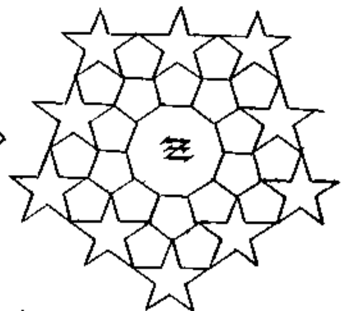
4. Now if we assume one of the three angles to be a trigon angle, we obtain a congruence with two dodecagon angles. This pattern can be continued without involving any different kind of meeting-point. The result in the plane is as shown in S. **S.**

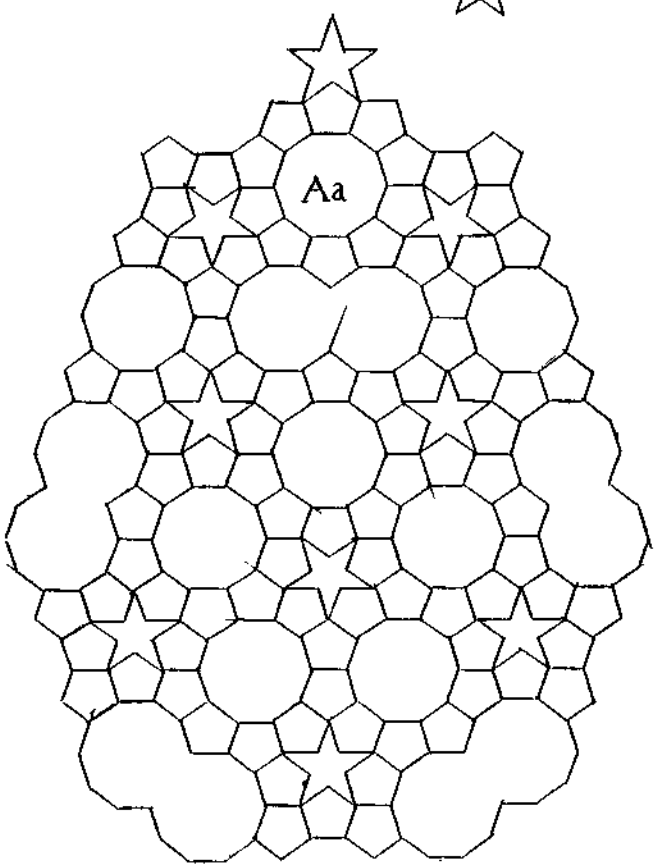
Here we must consider the star dodecagon, because its re-entrant angles are equal to a trigon angle, so that a dodecagon can be divided up into a star and twelve trigons. Therefore, five trigon angles and two points of two stars will fit together. The form, which can be continued, is as seen marked with the letter T **T.**

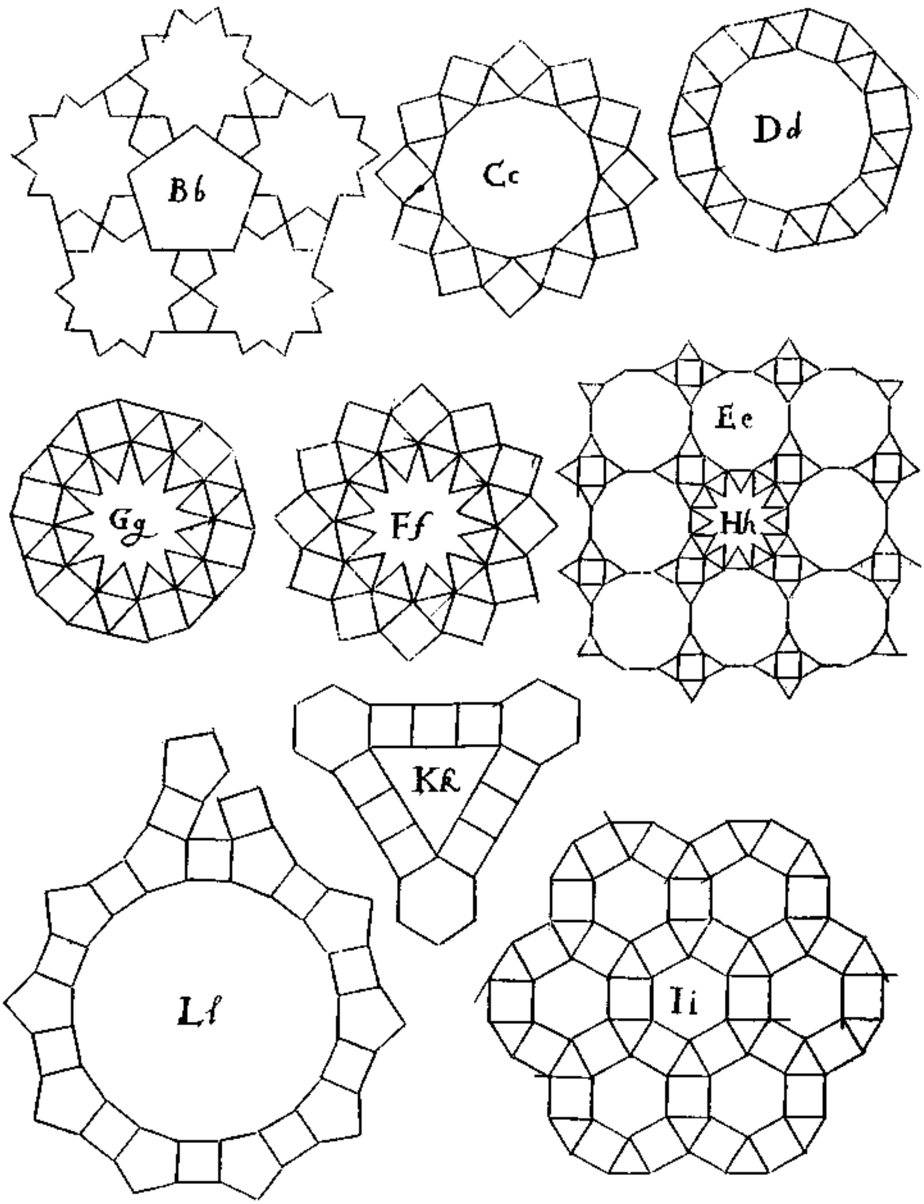
[18] For patterns L to X see Figure on our page 104. To extend M one must presumably first remove the triangles to left and right of the central row of squares. Coxeter (1975) has pointed out that pattern L is chiral, that is, it cannot be moved continuously into the position of its reflected image. The same is true of pattern N. For Kepler's lack of concern with chirality in regard to polygons, see note 133 on Book I above.

[19] *Duo* in the original edition (p. 52) and in KGW 6 (p. 72). Presumably this should be *tres,* since Kepler is considering sets of five angles.


Y


Zz


Aa


Bb
Cc
Dd
Ee
Gg
Ff
Hh
Kk
Ii
Ll

5. And if a tetragon angle is taken as one of the three, there is a congruence with two octagon angles, and this form too may be continued. It is seen marked
V. *with the letter V.*

Here we must consider the star octagon, because its re-entrant angles are equal to a tetragon angle, so that an octagon can be divided up into a star and eight right-angled triangles, two of which make a tetragon. And thus three tetragon angles and two points of two stars fill the plane. The mixed form is as shown
X,Y. *marked with the letter X, or otherwise, again mixed, marked Y.*

6. Having dealt with sets of three angles which include trigon angles and tetragon angles, if we now come to the pentagon angle we may take two of them, because together they come to more than two right angles; and a decagon angle fits into the space they leave. The decagon is encircled by ten pentagons, but this pattern
Z. *cannot be continued in its pure form. See the inner part of diagram Z.*

Here we must consider the star pentagon, since we can fit together three pentagon angles and one point of a star, because the re-entrant angle of the star takes one pentagon angle while, no less, the gap left by fitting together three pentagon angles takes the point of the star. See the outer part of the same
Z. *diagram* Z.

However, this pattern cannot be continued indefinitely, for the domain it builds up is unsociable[20] *and when it has added to its size a little it builds fortifications. You may see a different arrangement of these two forms marked with the letters Aa.*

If you really wish to continue the pattern, certain irregularities must be admitted, two decagons must be combined, two sides being removed from each of them. As the pattern is continued outwards five-cornered forms appear repeatedly: in the first and smallest of the five-cornered ranks there are five decagons with no intermediate irregularity, in the second and wider rank lines of single decagons lie between decagons joined in pairs, in the third rank the corners are taken by pairs of joined decagons and between two such pairs there lies a simple decagon,
Aa. *in the fourth rank*[21] *we again have simple decagons in the corners and on the side between them there are two more decagons, spaced at equal intervals, in the fifth rank the corners are marked by the tips of the outermost points of stars and the sides each contain two simple decagons between which there are two pairs of combined decagons. So as it progresses this five-cornered pattern continually introduces something new. The structure is very elaborate and intricate. See the diagram marked Aa.*

Here we must also consider the star decagon, whose re-entrant angle fits round the angle of a pentagon. In this way two of the points, each one three tenths of a right angle, join up with two pentagon angles to fill the plane around a point. This pattern takes in pentagons of a different size. It can be continued,

[20] That is, it will not form congruences. See Kepler's introduction to this Book
Aa. and Sections XII and XIII above.

[21] Kepler's diagram does not extend so far, nor does the diagram of the pattern which he supplied in a letter to Herwart von Hohenburg on 6 August 1599 (KGW 14, letter 130, pp. 21-41; see p. 33 for diagram).

but the continuation includes incomplete open decagons. The pattern is shown marked with the letters Bb. **Bb.**

We cannot take a single pentagon as one of the three plane figures which are to form a congruence, for its angle is six fifths of a right angle, by XXXIII of Book I, so the remaining two angles would be left with fourteen fifths of a right angle, that is each would be seven fifths, which is not the angle of any regular figure. Nor can we take two hexagons, for the remainder is also the angle of a hexagon and we shall obtain the form of congruence already described above, and we are now looking for structures involving two kinds of figure, not only one kind. As for the higher polygons, whose angles are greater than that of the hexagon, when two such angles are subtracted from four right angles the remainder is less than the angle of a hexagon; taking one angle from four right angles, what is left for the remaining two angles of the proposed congruence is less than two hexagon angles. We have already dealt with those figures whose angles are smaller than that of the hexagon, so we have dealt with all the ways of covering the plane with three figures at each meeting-point.

XX Proposition

There are four ways in which the plane can be filled by the congruence of plane angles of three kinds.

Here we cannot use three or more trigon angles at each meeting-point, for three trigon angles make two right angles, and so leave a gap which is less than the sum of the angles of the next two polygons, the tetragon and the pentagon. For the same reason we cannot employ two trigon angles with two tetragon angles, or with larger ones, since they do not leave enough space for the angle of the third kind of figure.

1. So if we have two trigon angles and one tetragon angle, the dodecagon angle will join up with them. However, this pattern cannot be continued. See letters Cc, Dd, and Ee, which show three forms, all belonging to the first case of the proposition. **Cc, Dd, Ee.**

Here, as above, we must consider the star dodecagon. For four trigon angles, one tetragon angle and one point of the star will fill the plane. See forms Ffi Gg, and Hh. **Ff, Gg, Hh.**

If a pentagon angle is joined up with two trigon angles the remainder will be incongruent, being twenty-two twenty-fifths of a right angle, for there is no angle of a regular figure which is eleven twenty-fifths. If one hexagon angle is added to two trigon angles the remainder will also be a hexagon angle, and the form will be one of those described above. So there are no more plane congruences involving two trigons.

So let there be one trigon angle. Three tetragon angles cannot be added to it, because then the sum is too large and there will not be enough space left for the angle of the third kind.

2. To one trigon angle let us add two tetragon angles. When we subtract the sum from four right angles the remainder is the angle of a hexagon. This pattern takes two forms: the one shown at Ii can be continued, the one shown at Kk **Ii, Kk.**

cannot be continued without additional figures. This is the second case of the proposition.

A single trigon angle cannot join up with two pentagon angles, because the gap left is fourteen fifteenths of a right angle, an angle not found in regular figures. Nor can a single trigon angle join up with one pentagon angle, because the gap left would be thirty-two fifteenths of a right angle and no regular figure has an angle of sixteen fifteenths. Nor can a trigon angle join up with one hexagon angle, because they come to two right angles, and no single angle is that size; and half of the gap is the size of the tetragon angle, which we have already dealt with. Nor can a trigon angle join up with one angle of a heptagon or of an octagon or of an enneagon, for the gap left for the third kind of angle would be forty twenty-firsts of a right angle or eleven sixteenths or sixteen ninths, none of which is the angle of a regular figure.

Now a trigon angle combined with a decagon angle leaves a gap of twenty-six fifteenths of a right angle, which is the angle of the pentehaedecagon. These figures do form a congruence, but a limited one. For the pentehaedecagon has an odd number of sides, so by XVII the meeting-points at the angles of the figure will not all be the same. Since the decagon has an even number of sides it can be surrounded by alternate trigons and pentekaedecagons, but two of the pentekaedecagons immediately run up against one another and prevent the pattern from being continued.

Further, a trigon cannot be joined with a hendecagon, for this would leave fifty-six thirty-thirds of a right angle, which is not the angle of any regular figure.

Next, a trigon angle taken together with a dodecagon angle leaves a gap the size of a decagon angle, a form we have already discussed.

If a trigon angle is subtracted from four right angles, the space left is not great enough to fit in angles of two other kinds, which together would come to more than two right angles.

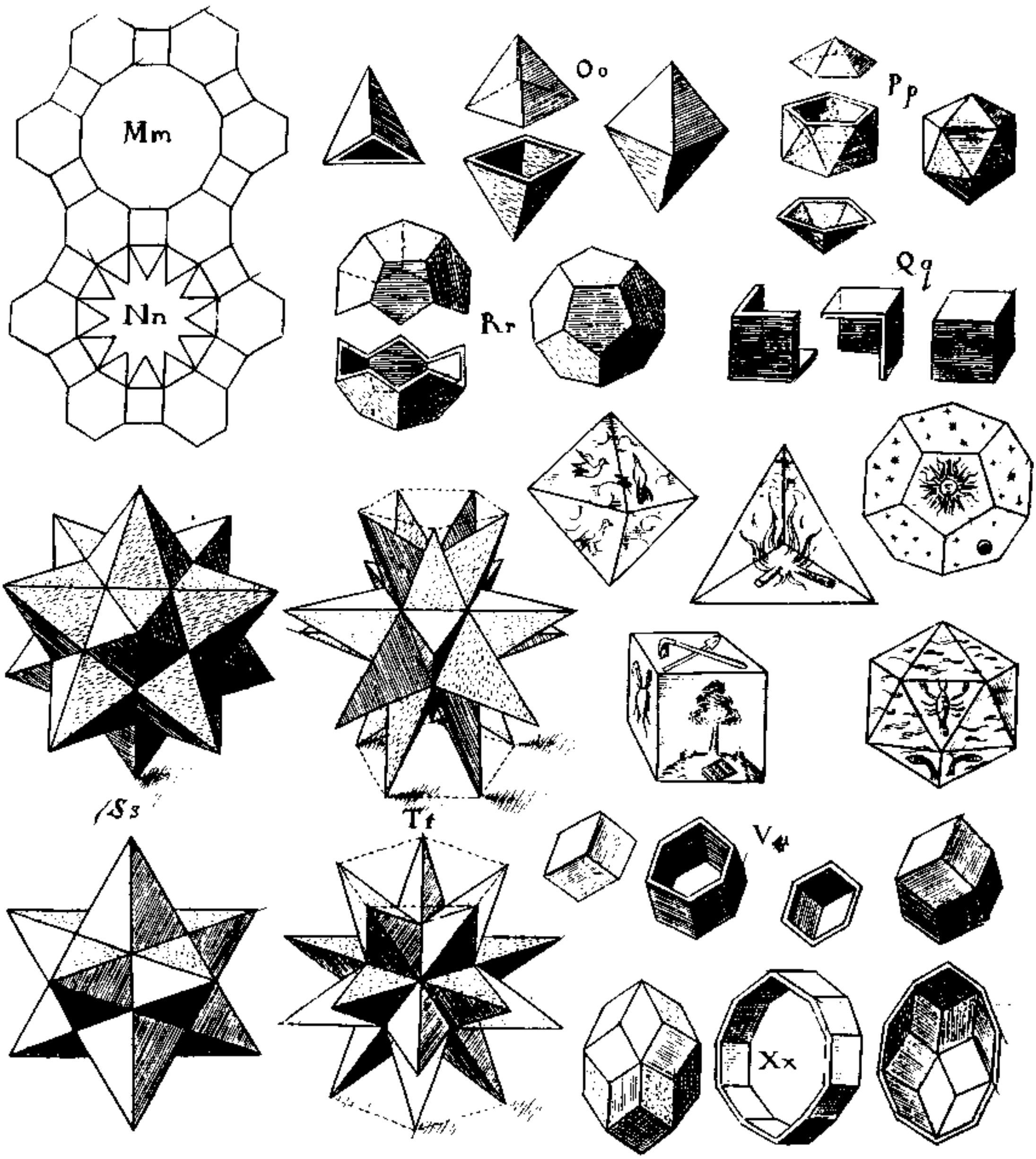
3. One tetragon angle joined up with one pentagon angle leaves a gap the size of an icosigon angle. An icosigon can therefore fit together with two such angles at every one of its own angles, forming a true congruence, but this pattern cannot be continued outwards. It is therefore an imperfect congruence. See figure LI. This is the third case of the proposition.

4. A tetragon angle joined up with a hexagon angle leaves a gap the size of a dodecagon angle. See figure Mm. This is the fourth and last case of the proposition.

Here we must consider the star decagon which can be filled out with twelve trigons. Four angles thus fit together to fill the space: two trigon angles, one tetragon angle, one hexagon angle, and a point of the star. See figure Nn.

If a tetragon angle is added to a heptagon angle it leaves a gap of eleven sevenths of a right angle, an angle not found in any regular figure. Added to an octagon angle, a tetragon angle leaves a gap the size of an octagon angle. We have discussed this form above. We have therefore dealt with all the cases involving a tetragon angle.

A pentagon angle together with a hexagon angle leaves a gap of twenty-two fifteenths of a right angle: taken together with a heptagon angle it leaves forty-


Mm
Nn
Oo
Pp
Qq
Rr
Ss
Tt
Vu
Xx

eight thirty-fifths; with an octagon angle thirteen tenths, while no such angle is to be found in any regular figure. And now [taking a pentagon angle with angles of figures having more sides than an octagon], the gap left starts to be less than an octagon angle, which is fifteen tenths of a right angle. But we have already dealt with congruences involving smaller angles. We have therefore dealt with all the cases involving a pentagon angle.

Three hexagon angles fill the plane, so a hexagon angle cannot be combined with two angles larger than itself. We have therefore dealt with all cases involving figures of three different kinds.

XXI Proposition

The plane cannot be filled by a congruence of the individual angles of plane figures of four or more kinds.

For the four smallest angles are those of the trigon, the tetragon, the pentagon, and the hexagon. And the first and last of these add up to two right angles, while the second of them is a right angle and the third is greater than a right angle, by one fifth of a right angle. Therefore, if they are joined up they come to more than four right angles. So by XVI they do not form a congruence. We should exceed four right angles by even more if larger angles were taken.

XXII Axiom

If two plane angles do not add up to more than a third one, they will not form a solid angle with it.

XXIII Proposition

Two plane angles of a figure with an odd number of sides will not come together with an angle of another kind to form a regular solid.

For by XVII the solid angles would not all be the same, which is not in accordance with definitions V to X.

XXIV Proposition

Three plane angles of figures of three different kinds, one kind having an odd number of sides, cannot come together in a perfect solid figure.

For, again by XVII, the solid angles would not all be the same, which is not in accordance with the definitions.

XXV Proposition

The most perfect regular congruences of plane figures to form a solid figure are five in number.

This is a scholium to the last proposition of the last book of Euclid.[22] *By XV of the present work we must start with three plane angles, and by XVI we must finish at six trigon angles, at four tetragon angles and at three hexagon angles, since, by XVIII, they add up to four right angles.*

Now three trigons, fitted together at one angle, make up less than four right angles in the plane, in fact they make up only two. When we make a solid angle by putting three trigons together the gap which remains can be filled by a fourth trigon. This gives the Tetrahedron or Pyramid.[22,] No. 2 in the following figure.

Four trigons, fitted together at one angle, make up eight thirds of a right angle, which is less than twelve thirds, or four right angles. Joining together the sides of the trigons we obtain a pyramid with an open four-sided base. Two such pyramids may be fitted together base to base to form a figure closed on all sides. This gives the Octahedron. Oo and no. 5 on the following page.

Five trigons, fitted together at one angle, make up ten thirds of a right angle, which is less than twelve thirds. Joining the sides together two by two round the common angle we obtain a pyramid with a five-sided base. Each of the angles around the base must eventually be made up of five plane angles, so each requires another three angles in addition to the two it already has. Thus the ten plane angles around the base require another fifteen, and fifteen angles will then point outwards in the opposite direction which adds up to thirty plane angles, that is tfie angles of ten trigons. These ten trigons make up a central zone or column, with a five-sided open end at top and bottom. Another pentahedral pyramid fits onto the open base, thus closing the figure all round. This gives the Icosahedron.

We have now dealt with all cases involving only trigons.

Three tetragon angles are three right angles, less than four right angles in the plane. Therefore they can be fitted together to form a solid angle. And when the tetragons are fitted together, they leave three gaps, and three angles of the plane stick out. So three more tetragons, fitted together to form a solid angle, will fit together with the first three, their points filling the gaps in the others and their gaps taking the points of the others. This gives the Hexahedron or Cube. Qq and no. 1 here.

Four tetragon angles are four right angles, therefore by XVI they do not form a solid angle. So we have dealt with all cases forming only tetragons.

Three pentagon angles are eighteen fifths of a plane right angle, which is less than twenty fifths, or four right angles. Therefore they can befitted together to form a solid angle. If we take one pentagon as a base and fit five others round it in this manner, we obtain a figure with gaps which are five pentagon angles and points which stick out which are also five plane pentagon angles. We can then construct another similar figure, in reverse, so that the five plane angles sticking out of the second figure will fit into the five gaps of the first one, and

[22] The scholium follows *Elements* XIII, 18, so Kepler presumably recognized that the Books XIV and XV found in many editions of Euclid's work were spurious (see Euclid trans. Heath, vol. Ill, pp. 507-508 for the scholium and pp. 512-520 on the spurious books).

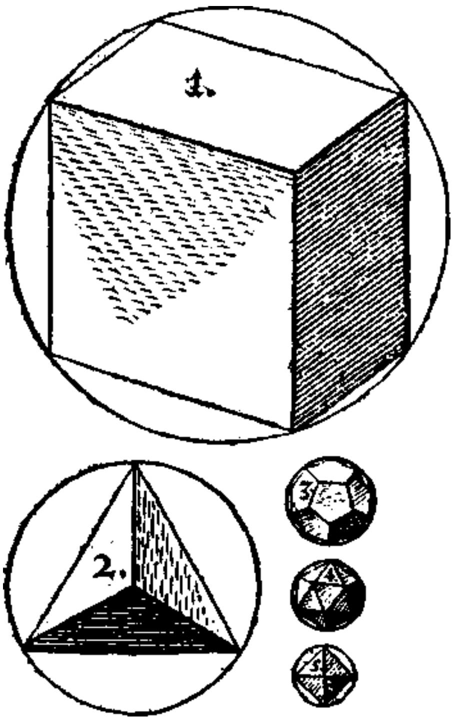
[23] See Figure on page 114.

Rr and no. 3 here.

vice versa. This produces the Dodecahedron. So we have dealt with all cases involving only pentagons, and thus with all those in which only one kind of figure is used: since, by XVI, three hexagons do not form a solid angle.

World figures.

These are the five bodies which the Pythagoreans and Plato, and Proclus, the commentator on Euclid, were accustomed to call the world figures, but, as I said in the introduction to Book I, it is not certain how they related these figures to the bodies of the world. The general persuasion, taken from Aristotle, is that since there were five such figures the philosophers related them to the five simple World Bodies, that is the elements: Fire, Air, Earth, and Water, and the Fifth Essence, or celestial matter, the characteristics of the figures being compared with the properties of the simple bodies. That the cube stands upright on a square base expresses stability, which is characteristic of terrestrial matter, whose weight tends down to the lowest point, while, as is commonly believed,[24] the whole globe of Earth is at rest at the center of the World. The octahedron, on the other hand, is viewed most appropriately suspended by opposite angles, as in a lathe, the square which lies exactly midway between these angles dividing the figure into two equal parts, just as a globe suspended by its poles is divided by a great circle. This is an image of mobility, as air is the most mobile of the elements, in speed and direction.



5. Octahedron symbol of air.

2. Tetrahedron symbol of fire.

4. Icosahedron symbol of water.

The tetrahedron's small number of faces is seen as signifying the dryness of fire, since dry things, by definition, keep within their own boundaries. The large number of faces of the icosahedron, on the other hand, is seen as signifying the wetness of water, since wetness, by definition, is held within the boundaries of other things. For a small number of faces indicates that a large number from another body will be associated with them. Furthermore, the plane trigon is proper to

[24] *cum . . . vulgo credatur.* The sense of *vulgo* may be intended to be in some degree pejorative.

That Kepler himself does not believe the polyhedra should be associated with the elements is clear from his ignoring such a theory in his discussion of the formation of snow in *De nive sexangula* (Prague, 1611). See also Field (1988).

the tetrahedron, since the complete tetrahedral figure is a solid trigon, while the same trigon is not proper to the icosahedron, but, rather, incidental to it, since the solid shape of the icosahedron is like a pentagon, not a trigon. Again, the tetrahedron's point, rising from one face, is seen as expressive of the penetrating and dividing power of fire, while the blunt quinquelinear angle of the icosahedron expresses the filling power of humours, that is their power to wet. The small thin tetrahedron shows the nature of fire; the large rounded mass of the icosahedron shows the nature of water, and as it were the shape of a drop. The tetrahedron has a very large surface and a very small body; the icosahedron has a bodily mass much greater than its surface: just as in fire it is the form that predominates and in water it is the matter.

The dodecahedron is left for the celestial body, having the same number of faces as the Zodiac has signs. It can be shown that it has the greatest volume of all the figures, just as the heavens enclose everything else.

5. Dodecahedron symbol of the heavens.

Although this analogy is acceptable, though not to Aristotle (who did not believe that the World had been created and thus could not recognize the power of these quantitative figures as archetypes, because without an architect there is no such power in them to make anything corporeal), yet it is acceptable to me and to all Christians, since our Faith holds that the World, which had no previous existence, was created by God in weight, measure, and number, that is in accordance with ideas coeternal with Him; although, I say, this sort of analogy is acceptable, yet framed in this manner it has no force of necessity; indeed, it admits of other interpretations, not only because certain properties are at variance within the analogy, but also because the dodecahedron and icosahedron correspond more closely with fire, and finally because the number of the elements and whether the Earth is at rest are matters much more open to dispute than is the number of the figures.

If the Pythagoreans held out against this theory, I do not blame Ramus, or Aristotle, for rejecting this disputed analogy. Twenty-four years ago I found out a very different relation between these five figures and the fabric of the world. I said in the introduction to Book I that I thought it likely that some of the ancients had been of this same opinion also, but had kept it secret, in the manner of their sect. For Copernican Astronomy, or the Astronomy of the ancient Pythagorean Aristarchus of Samos, describes the moving world as containing six spheres or paths surrounding the motionless body of the Sun, which is in the center, the spheres being separated from one another by large and unequal intervals. The outermost sphere is that of Saturn, the next that of Jupiter, then that of Mars, then that of the Earth and the Moon, then that of Venus and lastly that of Mercury, the innermost. Now we know that it is a fundamental property of these five figures that they can be inscribed within a spherical surface so that their angles are on the surface, and can also be circumscribed about a spherical surface so as to touch it at the centers of their faces; moreover, for any particular figure there is a particular interval between the two spheres defined in this way. Nothing seems more likely than that the five intervals between the six celestial spheres were taken by the Creator from the five figures, in an order such that the cube is to be imagined between the spheres of Saturn and Jupiter, the tetrahedron between those of Jupiter and Mars, the dodecahedron between those of Mars and the Earth, the icosahedron between those of the Earth and Venus, and the octahedron between those of Venus and Mercury.

See diagram in Book V, Chapter III.

This arrangement can be investigated numerically, and it has the force of necessity, not seeking anxiously for the number of the bodies but using the known number. In addition, it is so well constructed that no one has attacked it in these twenty-two years, but even the pupils of Ramus, that hot-headed scholar, the scourge of Euclid, even they have been drawn to it, and it now excites so much interest that mathematicians are calling for a second edition to be brought out.[25] But it is not the purpose of this second [i.e. present] book to go into details of this theory. The reader will find more about it below, in the fifth book, and something also in Book IV of the *Epitome Astronomiae*[26] where the true origin of these five solid figures is explained in metaphysical terms. For their origin is not really from the properties of their solid angles but, rather, the properties of the solid angles are a consequence of the origin of the figures, being by their very nature something that comes later.

XXVI Proposition

We may add to the most perfect regular congruences two further ones, each involving twelve star pentagons, and two semisolid congruences, of star octagons and star decagons.

Ss and below, 3ook V, Ch. I. Tt in the previous plate.

For star pentagons form solid figures closed on all sides and having spikes.[27] *One figure has twelve quinquelinear angles and the other has twenty trilinear angles. The former figure will stand up on three of its spikes, the latter on five at a time. The former looks handsomer if it stands upright on one spike, the latter sits more correctly when resting on five.*[29.] *The outsides of these solids do not show a regular face but instead an isosceles triangle containing a pentagon angle.*[29] *However, five such triangles always lie in a plane which has a five-*

[25] A second edition of the *Mysterium cosmographicum* was published in 1621.

[26] *Epitome astronomiae Copernicanae,* Book IV, Linz, 1620, Part I, pp. 45ff, KGW 7, pp. 267ff.

[27] Kepler's adjective is "aculeatus" (literally, prickly).

[28] In a letter written on 15/25 October 1618, Wilhelm Schickard (1592-1635) told Kepler that he had made two drawings of each of the spiked solids and Kepler might choose the ones he preferred (KGW 17, p. 279, letter 803, line 19). Kepler appears to have decided to use all the drawings.

[29] That is, the angle of the star pentagon.

cornered part covered by the body of the solid. The triangles surround this pentagon as if it were their heart, and together with it they make up the star pentagon, a figure called Witch's Foot in German, and by Paracelsus the sign of health. In structure, this body resembles its faces: in the face, a star pentagon, the sides of two triangles always lie in a straight line whose interior part not only forms the base of an exterior triangle but at the same time is the side of an inner five-cornered figure. Similarly, in the solid, individual isosceles triangles from five solid angles lie in a plane and the five-cornered innermost marrow or heart of the five triangles, or of the star, either forms the base of one of the protruding solid angles, or, in the other solid, is the base for five solid angles. These figures are so closely related the one to the dodecahedron and the other to the icosahedron that the latter two figures, particularly the dodecahedron, seem somehow truncated or maimed when compared to the figures with spikes.

The sides of the first and fourth points of star octagons and star decagons lie in a line, which passes through two intermediate points, and the stars can be fitted together with such sides joined two by two. The star octagons make a kind of cube, and the star decagons a kind of dodecahedron, figures which have not angles but ears, for when two of the plane angles are fitted together they must leave a gap, which cannot be closed.[50] *Therefore by XI the congruence is only semisolid.*

These solid and semisolid congruences are called most perfect because as solids they fit definition VI of this book. Their faces fit the definition of a perfect figure, which is the second definition of Book I, that is, they are secondary perfect figures. Nor is it absurd to call a semisolid congruence most perfect, because what we are concerned with is a congruence to which definition VI would apply if it could be completed, though definitions IX and X would not.

XXVII Proposition

Most perfect solid congruences are also formed by semiregular figures,[31] that is plane rhombi, and there are only two cases.

From twelve plane rhombi whose diagonals are in a particular ratio[32] *we may make a solid rhombus which has the shape of a honeycomb cell, that is, it has six sides and an end in the form of a trihedral angle. For if six rhombi are fitted together in such a way that obtuse angles meet obtuse angles and acute angles meet acute angles, then there will be three obtuse-angled gaps and acute-angled projections above, and the same will happen below. So three rhombi may be fitted in both above and below, with their obtuse angles fitted together to make a solid point and their remaining angles going into the gaps in the first figure, while the gaps left between the three new rhombi take the projecting points of the original figure.*

Vv and the smaller figure here.

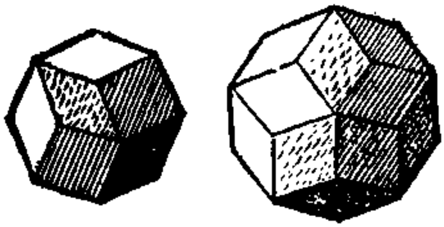
[30] In fact, the gaps can be filled with regular polygons: triangles for the eared cube and pentagons for the eared dodecahedron. See Badoureau (1881) and Field (1979a).

[31] See note 5 above.

[32] The ratio of the diagonals is $1:\sqrt{2}$. See also next note.

On this see also Def. V on p. 104 above.

Similarly, thirty plane rhombi, with a different ratio between their diagonals,^ make a solid triacontahedral rhombus. The rhombi are joined together at their acute angles, five by five, to give two solid angles pointing in opposite directions. There are gaps left where the obtuse angles meet. Each set of five gaps is then filled by the obtuse angles of a further five rhombi, and between these two shell-like figures we introduce a zone made of ten rhombi joined together. This is then joined to each shell.



Xx and the larger figure here.

We can show as follows that there are no further perfect congruences of rhombi. Two of the angles of a plane rhombus are acute and two obtuse, the sum of one acute angle and one obtuse one being two right angles. Further, it is not possible to put together more than three obtuse angles, since their sum would be greater than four right angles. By joining up only three acute angles one obtains something like a cube, a rhombic hexahedron, which has only two acute solid angles, the pair furthest away from one another. The other solid angles, in the middle of the body, lie closer together. The body does not satisfy definition VIII, which does not admit cases where only two solid angles lie on the same sphere. Moreover, each of the six obtuse solid angles is formed by two obtuse plane angles and one acute one, an irregularity which is once more contrary to the definitions. Therefore we may not fit together only three acute plane angles. But six angles, of six rhombi, will not fit together either. For if the individual acute angles are each two thirds of a right angle, the obtuse angles will be twice that size, that is four thirds. Thus both three obtuse angles and six acute angles will add up to four right angles, and neither the one set nor the other will form a solid angle, but instead the rhombi'will cover the plane continuously, as in G. If we now take smaller acute angles the corresponding obtuse angles will be larger than before and three of them will add up to more than four right angles. Therefore there are only two most perfect congruences of rhombi: one in which four acute angles of the rhombi make up a solid angle and another in which five do. However, the cube might be added to the list, as the first rhombic solid, for its faces also have four equal sides, as do those of the solid rhombi.

G on page 104.

XXVIII Proposition

There are thirteen solid congruences which are perfect in an inferior degree. From these thirteen we obtain the Archimedean solids.[34]

XIII Archimedeans.

33 The ratio of the diagonals is $1:\frac{1}{2}\sqrt{6-2\sqrt{5}}$. Both the rhombic solids are described, though without mention of the particular shapes of their faces, in *De nive sexangula* (Prague, 1611, p. 7; KGW 4, p. 266).

34 Pappus (Collection V, 19) gives cursory accounts of these solids (merely listing the numbers of faces they have of each shape) and ascribes their discovery to Archimedes. Kepler almost certainly read Pappus in Commandino's edition (Venice, 1588). See also Field (in press).

For congruences in this degree figures of different kinds are combined, thus by proposition XXI either two or three kinds of figure will be involved. Cases involving two kinds either will or will not include trigons.

Thus, trigons and tetragons will make three solids which satisfy definition IX. This definition rules out three ways of constructing a solid angle, namely by using one tetragon angle and either one or three trigon angles or by using two tetragon angles and one trigon angle. For in the first case the congruence includes only one tetragon and we obtain half an octahedron, a figure whose solid angles are not all alike, while in the second case we have only two tetragons[35] *and in the third only two trigons.*[36] *So all these congruences are imperfect, by definition X. There remain the following methods of constructing a solid angle from plane ones. First, by using four trigon angles and one tetragon angle. For they add up to less than four right angles. Thus six tetragons and thirty-two (that is twenty and twelve) trigons fit together to make a triacontaoctahedral figure which I call a snub cube.*[37] *It is shown in the diagram below, numbered 12.* Oo. I Snub cube.

Five trigon angles and one tetragon angle are more than four right angles, whereas to form a solid angle they would need to be less than four right angles, by XVI. The same is true of four trigon angles and two tetragon angles. In fact, three trigon angles and two tetragon angles make four right angles.

Second, two trigon angles and two tetragon angles are less than four right angles. Thus eight trigons and six tetragons fit together to form a tessareskaedecahedron, which I call a cuboctahedron[36] *It is shown here with the number eight. Two trigon angles and three tetragon angles are more than four right angles.* II Cuboctahedron.

Third, one trigon angle and three tetragon angles come to less than four right angles. Therefore eight triangles and eighteen (that is, twelve and six) squares join up to make an icosihexahedron, which I call a truncated cuboctahedral rhombus or a rhombicuboctahedron.[39] *It is shown on this page, numbered 10.* Ill Rhombicuboctahedron.

[35] The solid is an antiprism (type A in Figure on page 104) with a square base.

[36] The solid is a prism (type B in Figure on page 104) with a triangular base.

[37] *Cubus simus.* In Kepler's unfinished treatise on geometry (1628-1630, Pulkova MS XXII, printed in KOF VIII, 1, pp. 174ff.) the snub cube and the snub dodecahedron (number IV below) are associated with the "mixed" solids, the cuboctahedron and the icosidodecahedron (numbers II and V below; see KOF VIII, 1, p. 182). The designation "mixed" and the name "icosidodecahedron" are both apparently derived from Foix de Candale, *Demixtis et compositis regularibussolidis,* published with Foix de Candale's edition of the *Elements* (Paris, 1566). Foix de Candale's name for the cuboctahedron is "exoctohedron."

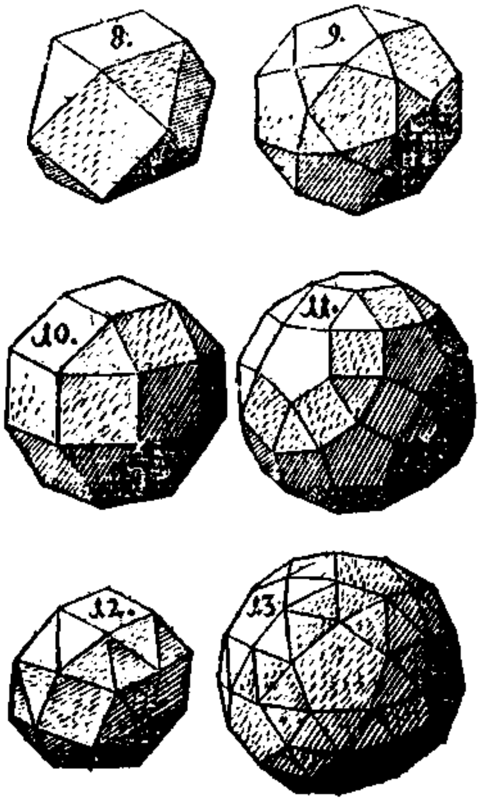
[38] See previous note.

[39] *Sectus rhombus cuboctaedricus, rhombicuboctaedron.* The rhombicuboctahedron can be obtained from the solid formed by truncation of the cuboctahedron by distorting it in such a way that the rectangular faces produced by the truncation become square. For the undistorted solid, see illustrations from Wentzel Jamnitzer, *Perspectiva corporum regularium* (Nuremberg, 1568), reproduced in Field (1979a), Figure 15, and Field (1988), Figure A4.8 (bottom left). For the discovery of this solid, found in Pacioli (1509), see Field (in press). Compare method of obtaining the Archimedean truncated cuboctahedron, see note 49 below.

In these three figures we have tetragons combined with trigons. In what follows we shall combine each of them separately with pentagons.

IV Snub Dodecahedron.

Five trigon angles will not combine with a pentagon angle, since they will not even combine with a tetragon angle, which is smaller. Four trigon angles and one pentagon angle make less than four right angles, and eighty (that is twenty and sixty) trigons will fit together with twelve pentagons to make an enenecontakaedyhedron, which I call a snub dodecahedron.[40] *It is shown here numbered 13. In this series of snub figures the icosahedron could make a third, since it is like a snub tetrahedron.*[41]



If you combine three trigon angles with one pentagon angle the result is as described above, namely that the solid formed includes only two pentagons; and if you combine two trigon angles with one pentagon angle the solid includes only one pentagon. The former case gives a zone or central column and the latter gives a pyramid, both parts of an icosahedron. The solid angles of the second body are not all the same, since one is surrounded by five trigon angles, as in an icosahedron. We have now dealt with all the cases involving only one pentagon angle.

Three trigon angles with two pentagon angles make more than four right angles. So we have dealt with all the cases in which three trigon angles are combined with pentagon angles.

V Icosido-decahedron.

Two trigon angles with two pentagon angles make less than four right angles. Thus twenty trigons and twelve pentagons fit together to make a triacontakaedyhedron, which I call an icosidodecahedron.[42] *It is shown here numbered 9. Since we have already rejected the case in which two trigon angles are joined up with one pentagon angle we have now dealt with all the cases involving two trigons.*

One trigon angle added to three pentagon angles makes more than four right angles, and if it is joined up with two pentagon angles it cannot make a regular solid, by XXIII, since the pentagon has an odd number of sides.

[40] *Dodecaedron simum.* See note 37 above.

[41] This is explained more fully in the treatise on geometry referred to in note 37 above. The "mixed solid" to which the snub tetrahedron is related is the octahedron (seen as a tetra-tetra-hedron).

[42] *Icosidodecahedron* (sic). See note 37 above.

So we have now dealt with all the cases involving pentagons combined with trigons.

Four trigon angles with one hexagon angle, and two trigon angles with two hexagon angles fill the plane around a point; three trigon angles with two hexagon angles are greater than four right angles, and with only one hexagon angle they give a figure which contains only two hexagons. So we must reject cases involving three trigon angles. Two trigon angles are equal to one hexagon angle, so this case is also rejected, by XXII. It remains to unite one trigon angle with two hexagon angles. Thus four trigons and four hexagons fit together to make an octahedron, which I call a truncated tetrahedron.[43] *It is shown as number 2 on the next page.* VI Truncated tetrahedron.

Four trigon angles with one heptagon angle, or a larger angle, come to more than four right angles, so we need not discuss cases involving four trigon angles, nor cases involving three, for reasons already given. In fact, two trigon angles with two plane angles larger than those of the hexagon come to more than four right angles, so we do not need to discuss cases involving two trigon angles joined with two plane angles of a figure larger than the hexagon, nor cases involving two trigon angles joined with one plane angle of a larger figure, because such an angle is larger than two trigon angles, so the case is rejected by axiom XXII. It remains for us to examine the case in which one trigon angle is united with two plane angles of a figure larger than a hexagon. The case in which these are two heptagon angles is rejected by XXIII, as are all the cases involving two angles of a figure with an odd number of sides. With two octagon angles we obtain a solid in which eight trigons join up with six octagons to make a tessarakaedecahedron, which I call a truncated cube.[44] *There is a diagram of it numbered 1 on the following page. With two decagon angles we obtain a solid in which twenty trigons join up with twelve decagons to make a triacontakaedyhedron which I call a truncated dodecahedron.*[45] *This is shown below as number 3. With two dodecagon angles the plane is filled, so no solid angle can be made with these or any still larger angles. We have thus entirely finished with cases involving trigons together with any other single kind of figure.* VII Truncated cube. VIII Truncated dodecahedron.

Since the two kinds of plane figure will no longer include trigons the smallest figure involved will now be the tetragon. Three tetragon angles with one larger angle come to more than four right angles, and by definition IX we know we cannot combine two tetragon angles with one larger angle, since only two of the larger figures will occur in the resultant solid. The case of one tetragon angle combined with two pentagon angles is rejected, by XXIII, but one tetragon angle will go with two hexagon angles, and six tetragons and eight hexagons will fit together to make a tessarakaedecahedron which I call a truncated octahedron.[46] *It is shown as number 5 in the diagram below. The case of one* IX Truncated octahedron.

[4S] *Tetraedron truncum.* Kepler's names "truncated tetrahedron," "truncated cube," etc. continue to be used, despite the fact that in modern terms the solids are seen as .Krai-truncations of the tetrahedron, the cube, etc.

[44] *Cubus truncus.* See previous note.

[45] *Dodecaedron truncum.* See note 43 above.

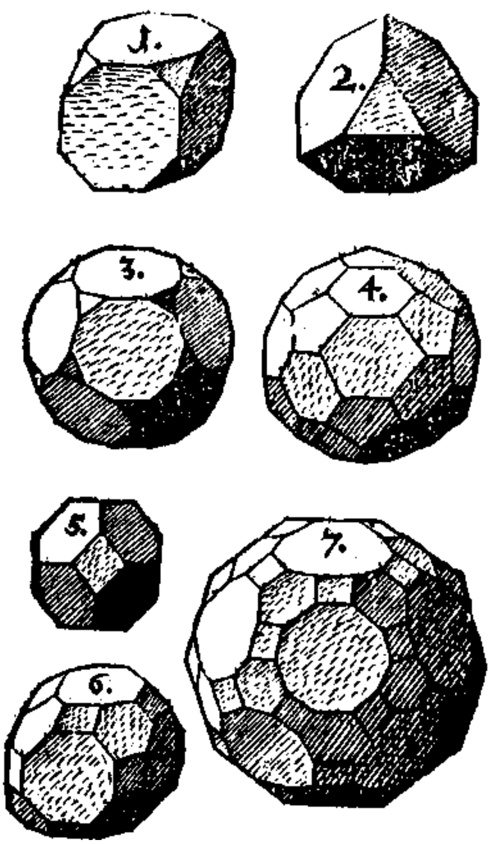
[46] *Octaedron truncum.* See note 43 above.

tetragon angle combined with two heptagon angles is rejected because the heptagon has an odd number of sides, that is by XXIII. With two octagon angles the plane is filled. With larger angles the sum exceeds four right angles and no solid angle can be formed. So we have dealt with all the cases involving the tetragon, since there must be only two kinds of plane figure.

Two pentagon angles combined with one hexagon angle or the angle of any other figure will not form a congruence, by XXIII, so these cases must be rejected, just as we earlier rejected the cases involving a trigon angle or a tetragon angle combined with two pentagon angles. Moreover, two pentagon angles combined with one decagon angle cover the plane, so neither with this angle nor with a larger one will they form a solid angle.

X Truncated icosahedron.

Now one pentagon angle with two hexagon angles comes to less than four right angles, and twelve pentagons and twenty hexagons will fit together to make a triacontakaedyhedron, which I call a truncated icosahedron.[41] *It is shown numbered 4. We cannot expect any more congruences from the pentagon. For one pentagon angle combined with two heptagon angles is already larger than four right angles.*



One hexagon angle with two others fills the plane, and with two larger angles the sum exceeds four right angles. So we have now dealt with all the cases in which two kinds of figure are combined.

Turning to cases in which three kinds of face may fit together to make a solid angle, we first note that two plane angles, one from a tetragon and one from a pentagon, add up to more than two right angles, and larger angles add up to even more, so since three trigon angles come to two right angles, it is clear that the two plane angles we have mentioned will not fit together with three trigon angles, for the total sum would be more than four right angles. The cases in which two trigon angles are combined with one tetragon angle and one pentagon angle, or, instead of the pentagon angle, a hexagon angle or some larger one, all these cases are to be rejected, by proposition XXIII, because they would require the trigon, which has an odd number of sides,

[47] *Icosihedron* (sic) *truncum.* See note 43 above.

to be surrounded by tetragons and either pentagons or, instead of the pentagons, hexagons, etc.

One trigon angle, two tetragon angles and one pentagon angle add up to less than four right angles, and twenty trigons, thirty tetragons and twelve pentagons will fit together to make a hexacontadyhedron which I call a rhombicosidodecahedron or a truncated icosidodecahedral rhombus.[48] *It is shown as number 11 on the previous page.* XI Rhombicosidodecahedron.

One trigon angle and two tetragon angles combined with one hexagon angle come to four right angles; with a larger angle they come to more; so they do not form a solid angle. Let us therefore dismiss cases involving two tetragon angles.

One trigon angle, one tetragon angle and two pentagon angles add up to more than four right angles, and they add up to even more with two larger angles. So we have finished with cases in which four plane angles are put together to form a solid angle, and also with cases in which the trigon angle is one of the three kinds of angle involved. For the case of one trigon angle and one tetragon angle and either one pentagon angle or any other angle is to be rejected, by XXIV, since the trigon has an odd number of sides.

In fact, since we are now combining only three plane angles, no figure with an odd number of sides can be admitted, by XXIV again.

A tetragon angle with a hexagon angle and an octagon angle, the smallest admissible angles, comes to less than four right angles; and twelve tetragons, eight hexagons and six octagons will fit together to make an icosihexahedron which I call a truncated cuboctahedron: not because it can be formed by truncation but because it is like a cuboctahedron that has been truncated.[49] *It is shown numbered 6.* XII Truncated cuboctahedron.

A tetragon angle with a hexagon angle and a decagon angle comes to less than four right angles; and thirty tetragons, twenty hexagons and twelve decagons fit together to make a hexacontadyhedron which I call a truncated icosidodecahedron, for reasons similar to those given in the previous case.[50] *It is shown numbered 7.* XIII Truncated icosidodecahedron.

If we replace the decagon angle by a dodecagon angle the sum is four right angles and we cannot make a solid angle; also, if we replace the hexagon angle by an octagon angle and take as our third angle any angle larger than an octagon angle we have more than four right angles; nor is the sum less if we set aside the tetragon angle and instead join up three angles from larger figures with an even number of sides. Therefore the whole family of Archimedean solids numbers thirteen, as was to be shown.

[48] *Rhombicosidodecaedron. Sectus rhombus icosidodecaedricus.*

[49] *Cuboctaedron truncum.* The Archimedean solid may be obtained by distorting the true semitruncated cuboctahedron in such a way that the rectangular faces produced by the truncation become square. For the undistorted solid, see illustrations from Wentzel Jamnitzer, *Perspectiva corporum regularium* (Nuremberg, 1568), reproduced in Field (1979a), Figure 15, and Field (1988), Figure A4.8 (bottom right). Compare method of obtaining the Archimedean rhombicuboctahedron, see note 39 above.

[50] *Icosidodecaedron truncum.* See previous note.

XXIX Conclusion

There are in all twelve figures which will form congruences, eight basic or primary figures and four augmented or star figures.

1. Trigon	7. Dodecagon
2. Tetragon	8. Icosigon
3. Pentagon	9. Star pentagon
4. Hexagon	10. Star octagon
5. Octagon	11. Star decagon
6. Decagon	12. Star dodecagon

The degrees of congruence are distinct. The trigon and the tetragon are of the first degree, because they form congruences in space as well as in the plane, both among themselves, with figures all of one kind, and also when combined with other figures.

The pentagon and its star are of the second degree. For in space they will form congruences among themselves, with figures all of one kind, and in the plane they come to one another's aid, but the pentagon is the more powerful of the two because it will also form congruences with some other figures, both in the plane and in space.

The hexagon is of the third degree, because figures of this kind form congruences in the plane, and in combination with other figures will form congruences both in the plane and in space.

The fourth degree is taken by the octagon and the decagon and their stars. For the basic figures will form solid congruences with some other figures and the stars will form congruences with figures all of one kind at least to a limited extent.[51] In the plane all four figures form congruences with others, the octagonal figures doing so in more ways and more perfectly.

The fifth degree is that of the dodecagon and its star, because in space they do not form any congruences at all whereas in the plane they combine with other figures to form many different congruences. In space it is only their size which prevents them forming congruences. As far as congruence in the plane is concerned this group should be of the fourth degree.

The icosigon is of the last degree, because this figure will form congruences only in the plane and then only when combined with other figures, and, moreover, these congruences are imperfect.

So if we consider only the plane, the order of the figures will be the following: 1. Hexagon, 2. Tetragon, 3. Trigon, 4. Dodecagon, 5. Its star, 6. Octagon, 7. Its star, 8. Pentagon, 9. Its star, 10. Decagon, 11. Its star, 12. Icosigon.

All other figures are incapable of forming congruences, though the figure that comes closest to doing so is the pentekaedecagon, because it begins to form congruences with other figures in the plane;

[51] The semisolid congruences are described in Section XXVII above.

but it is excluded, by XX, because, unlike the icosigon, it cannot be surrounded at all its angles in the same way. After that comes the figure with sixteen sides and others like it, which do not form plane congruences with other regular figures because their angles are too large. But the heptagon and similar figures do not form congruences for a quite different reason, namely because neither whole angles nor aliquot parts of an angle of such a figure are able to form congruences with other regular figures.

So congruence can be divided into three demonstrably distinct classes: the octagon class, the decagon class, and the icosigon class, together with a fourth, spurious, class in which there is no congruence. These classes will find their application in the choice of Aspects in Book IV.

XXX Conclusion

From this we see that there is a genuine difference between construction and congruence in respect of the width of the classes they form.

For, 1. The degrees of proper construction extend to infinity from the octagon, decagon, and dodecagon to include all figures that can be obtained by successive doubling of the number of sides; congruence is confined to the degrees of the octagon, the icosigon, and the dodecagon. 2. In respect of construction and knowledge the pentagon and its star are less noble than the dodecagon; in respect of congruence in space they are much nobler. 3. In construction and knowledge the octagon ranks lower than the pentagon but takes precedence over it in congruence. 4. The hekkaedecagon was higher placed than the icosigon for construction, yet the former will not form congruences whereas the latter will, to a limited extent. 5. But the pentekaedecagon shows a pleasing uniformity of properties in these two respects: since it has no proper construction but only an accidental one and it will not form any complete congruences but only the beginning of a congruence which does not surround the whole figure. These properties are to be taken into account below in Book III in connection with the origin and use of the semitone. Page 250

End of Book II

Book III follows, with new font for the letters of the alphabet and a new start to the numbering of the pages, because it was with this book that the printing began.