BOOK III

OF THE HARMONY OF THE WORLD

by

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ON THE ORIGIN OF THE HARMONIC PROPORTIONS, AND ON THE NATURE AND DIFFERENCES OF THOSE THINGS WHICH ARE CONCERNED WITH MELODY.

Proclus Diadochus, Commentary on Book I of the Elements of Euclid, Book I:[1]

Although philosophy has embraced many skills, and so has mathematics, yet he writes the following about one part of the latter, called harmony, and about Numbers (believed to be the principles of harmonies).

Mental endeavor is the preparation for theology. For those features which to the uninitiated in the truth of divine matters seem difficult to grasp and lofty are by mathematical reasoning shown to be trustworthy, manifest and uncontroversial, by means of certain images. For they show proof of the supernatural properties in numbers; and they make clear the powers of the intelligible forms in reasoning. Thus Plato teaches us many remarkable things about the nature of the gods through the appearance of mathematical things; and the Pythagorean philosophy disguises its teaching on divine matters with these, so to speak, veils. For of this kind is the whole of that sacred

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[1] The Greek text has been published in Gottfried Friedlein (1873), pp. 22 and 24. The two paragraphs of the quotation are taken from different parts of the chapter on the utility of mathematics. There is an English translation of Proclus' commentary by G.R. Morrow (1970).

writing,[2] both Philolaus on the *Bacchae*,[3] and the whole Pythagorean system of teaching about God.

Again, it perfects us in moral philosophy, implanting in our behavior order, propriety and harmony in social relations. It also informs us what figures, what songs, what motions are appropriate to virtue; and also the teaching by which the Athenian[4] would have those who will pay attention to moral virtues from their youth cultivated and perfected. Furthermore he makes plain the proportions of numbers which are associated with the virtues, some in arithmetic, some in geometry and others in harmony; and he shows the excesses and deficiencies of the vices. By all of these we are guided to the middle way in behavior and in morals.

[2] This seems to be a reference to the ἱερός λόγος cited by Iamblichus and others. For references, see Morrow (1970), p. 19.

[3] For the fragments of Philolaus on the *Bacchae*, see H. Diels and M. Kranz (1972–74), vol. 1, Philolaus B. Fragmente pp. 17–19.

[4] See Plato, *Laws*, 672–673. The "Athenian stranger" is Plato's mouthpiece in the dialogue.

ON THE ORIGIN OF THE HARMONIC PROPORTIONS, AND ON THE NATURE AND DIFFERENCES OF THOSE THINGS WHICH ARE CONCERNED WITH MELODY.

So far, following the natural order, we have spoken first of Regular Plane Figures, and next we passed on to their congruences.

In what follows it will not be right to depart at all from the natural method, so that the learning of the human mind, which quite often uses a different route, may be given all the more assistance. For what the nature of the subject requires is that we should now thirdly expound in the abstract those proportions which occur between a circle and a part cut off on any side,[5] and the other kinds of cases which arise from the combination and separation of such proportions; then fourthly that we should pass to the operations of the world, which either God himself the Creator has adjusted to proportions of that kind,[6] or Sublunary Nature applies daily according to the rule of those proportions in the angles between the stellar rays;[7] and finally we should add human music, showing how the human mind, shaping our judgement of what we hear, by its natural instinct imitates the Creator by showing delight and approval for the same proportions in notes which have pleased God in the adjustment of the celestial motions. For it is indeed difficult to abstract mentally the distinctions, types, and modes of the harmonic proportions from musical notes and sounds, since the only vocabulary which comes to our aid, as is necessary to expound matters, is the musical one. Therefore in this Book we shall have to combine the third section with the fifth and last one,[8] and we shall not only have to speak of harmonic proportions in the abstract, but also to deal in anticipation with this human

[5] That is, by a side of a regular polygon inscribed in the circle.

[6] The celestial harmonies, whose treatment is deferred to Book V.

[7] The explanation of the dependence of the weather on the aspects, which will be described in Book IV.

[8] That is, the explanation of the musical harmonies in terms of the division of the circle.

imitation of the creation in melody, though it has been specified in the title of the Book that it should deal at that point with the work of the creation of the heavens, which must be postponed to the last section on account of its sublime and incredible nature. So much for the order of treatment.

Now in order to throw more light on the contraries which are opposed to each other, it is desirable to inaugurate this dissertation on human melody by recalling what the ancients have said about the origin of consonances.

Certainly, just as it is ordained in all human affairs that in those things which are bestowed on us by nature, use precedes understanding of causes, similarly as far as melody is concerned it happened to the human race that from its very beginning it used without speculating or knowing about their causes the same rhythms and intervals between notes as we commonly use today, in the chanting of melodies, not only in churches and in choirs of musicians, but everywhere without applying any art, even at crossroads and in the fields.

This antiquity of melody is apparent from the first book, of Genesis.[9] For great must the delight in the melody of the human voice have been (when I say delight, I mean the harmonious and melodic intervals) which moved Jubal,[10] eighth in line from Adam, to learn and teach how to imitate the melodies of men with inanimate instruments. Unless I am mistaken, this Jubal is the Apollo, by a slight change of letters, who defeated his brother Jabel, the originator of cattle breeding, whose joy was in the shepherd's pipe (and who was believed by the Greeks to be the god Pan), by the clear ringing of the lyre which he had invented, having borrowed the material for the strings from his brother Tubalcain (and let him be Vulcan for us, by a play on the name).

Yet however ancient be the pattern of human melody, made up of consonant or melodic intervals, yet the causes of the intervals have remained unknown to men—so much so that before Pythagoras they were not even sought; and after they have been sought for two thousand years, I shall be the first, unless I am mistaken, to reveal them with such accuracy.[11]

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[9] Genesis 4, vv. 20–22.

[10] By his free interpretation of Scripture, Kepler here seeks to identify the biblical and mythological accounts of the origin of music, comparing Jubal with Apollo, Jabel with Pan, and Tubalcain with Vulcan. He projects the mythological contest between Apollo (the god of poetry and music) and Pan (the god of flocks and shepherds, who is usually depicted carrying a pipe of seven reeds called the syrinx) on to the biblical history. According to the mythology, Apollo won, using the kithara, for which Vulcan supplied the strings. According to the music theorists, however, Jabel was not connected with the shepherd's pipe but it was Jubal who invented both string and wind instruments. See Dickreiter (1973), p. 148 and Barker (1984), p. 46.

[11] Before he received from Herwart von Hohenburg in July 1600 a copy of Ptolemy's *Harmonica* in the translation of Antonius Gogavinus, Kepler's principal source

It is said indeed that Pythagoras[12] was the first, when he was passing through a smithy, and had noticed that the sounds of the hammers were in harmony, to realize that the difference in the sounds depended on the size of the hammers, in such a way that the big ones gave out low sounds, and the little ones high sounds. Now as a proportion is properly speaking observed between sizes, he measured the hammers, and readily perceived the proportions at which consonant or dissonant intervals occurred, and melodic or unmelodic intervals occurred between notes. Indeed he passed at once from the hammers to the length of strings,[13] where the ear indicates more exactly what fractions of the string are consonant with the whole, and which are dissonant with it.

Having discovered definite proportions, or "the fact that," it remained to track down the causes as well, or "the reason why"[14] some proportions marked out melodic, pleasant, and consonant intervals between notes, and other proportions those which are dissonant, abhorrent to the ear, and strange. And in the course of two thousand years the opinion has been reached that the causes are to be looked for in the properties of the proportions themselves, as they are con-

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for his knowledge of ancient music, which in this book he describes and compares with his own, was Boethius, *De institutione musica*, published in Venice in 1492. Kepler also used a number of other books. One of these, the *Elementa musica* of Aristoxenus, the most famous and influential music theorist of antiquity, was included by Gogavinus in his edition of Ptolemy. Another book studied by Kepler was the pseudo-Euclidean *Harmonica* that Konrad Dasypodius appended to his edition of Euclid's works, published in Strasbourg in 1571. The *Commentarii in somnium Scipionis* of Macrobius, published in 1472, was used by Kepler for the clarification of Ptolemy's assignment of harmonies to the planets, on which he did not have the complete text of Ptolemy. (See KOF, vol. 5, p. 410). Yet another work used by Kepler was Jordanus de Nemore's *Arithmetica, musica, epitome in libros arithmeticos Boetii*, published by J. Faber Stapulensis in 1496.

As he was dissatisfied with the Gogavinus translation of Ptolemy's *Harmonica*, Kepler requested a loan copy of the Greek manuscript from his friend Herwart, the Bavarian chancellor. This he received in 1607 together with a copy of Porphyry's commentary, which goes up to chapter 7 of Book II (KGW 15, p. 408). Kepler's own translation of Ptolemy's *Harmonica*, Book III, chapters 3–16, which he began to prepare as soon as he received the manuscript and intended to publish in the *Harmonice mundi*, was first published in the nineteenth century by Frisch (KOF vol. 5, pp. 335–412).

Among the contemporary works used by Kepler were Seth Calvisius' *Melopoeia seu melodiae condendae ratio* (1592), Giovanni Maria Artusi's *L'Arte del contraponto* (1586–9) and Vincenzo Galilei's *Dialogo della musica antica et della moderna* (1581). Gioseffo Zarlino's *Istitutioni harmoniche* (1558) may have been known to Kepler only indirectly through Calvisius. Zarlino's work contains the essentials of Ptolemy's theory, while Galilei defended the older Pythagorean theory.

[12] This legend is related by Boethius, *De institutione musica*, Book I, Chapter 10.

[13] According to Boethius, *De institutione musica*, Book I, Chapter 11, he experimented also with pipes.

[14] The distinction between the question of fact and the question of cause, emphasized here by Kepler in his use of the Greek terms τὸ ὅτι and τὸ δι'ὅτι, is an important principle of Aristotelian logic. See, for example, Aristotle, *Posterior analytics*, 89 b 23–35.

Pythagorean philosophy on the power of numbers.

tained within the boundaries of a discrete quantity, that is to say of Numbers.[15] For the Pythagoreans saw that perfect harmonies were established if cords under equal tensions have their lengths in double proportion, or triple, or quadruple, as between the numbers 1 and 2 or 1 and 3 or 1 and 4. Such proportions are called in arithmetic multiple. Further, slightly more imperfect consonances occurred between the strings which make up the sesquialterate proportion, the Hemiholia, and the sesquitertiate, the Epitriton, that is between the numbers 2 and 3 and 3 and 4. These two proportions combined make the double proportion, as between the numbers 2 and 4 or 1 and 2; but the smaller proportion, between the numbers 3 and 4, divided into the greater, that between 2 and 3, left the proportion of one to one and an eighth, that is between 8 and 9. And this, they discovered, was the size of the interval of a tone, the commonest of all in melody. But the number 8 is the cube of 2, and the number 9 is the square of 3. Then the following numbers were already before them: 1, 2, 3, 4, 8, 9. However since Unity is the same as its square and its cube, whereas the binary had as its square 4 and as its cube 8, to the ternary they also added its cube 27 as well as its square 9, because they supposed that it was right always go as far as the cubes on account of the fact that the whole world and everything that gives notes consisted not of empty surfaces but of solid bodies. Eventually from that beginning such a strong opinion grew up about these numbers, on account of the fact that they were Primes, and their squares and their cubes, that the Pythagoreans resolved that the whole of Philosophy should be composed of them. For Unity represented for them Idea and Mind and Form, because just as Unity is indivisible and remains the same when it is squared or cubed, so the Ideas also were irreducible and universal and always the same. Therefore they made Unity the symbol of the nature of Identity, but the other numbers the symbols of the nature of otherness. Then the Binary signified otherness and matter, because the former admits of division, and so does the latter; and as the Binary squared becomes 4, and cubed becomes 8, which are numbers distinct from 2, so matter can be unstable and multiform. On the other hand, the Binary also signified Soul, because although Mind is immobile, or takes joy in uniform, that is circular motion, Soul on the contrary receives multiple motions from Body, and is more amenable to rectilinear motions, which are differentiated in six ways. Lastly the Ternary denoted for them Substance, which is made up of Form and Matter, just as 3 is made up of 2 and 1, and because bodies in the real world have the same number of dimensions as the Ternary has Unities.[16]

1.
2. 3.
4. 6. 9.
8. 12. 18. 27.

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[15] The Pythagorean philosophy of numbers is described by Aristotle in *Metaphysics*, 985 b 24–986 b 8. The principal source for the musical theory described here is the construction of the world-soul in Plato's *Timaeus*, 34C–36E.

[16] See, for example, Aristotle, *Metaphysics*, 987 b 20–30.

Soul to the Pythagoreans is Number and Harmony.

Nor were the numbers symbols only of the three basic principles, but moreover Soul itself was made up of these very numbers, and of all their proportions, and the subdivision of their proportions into sesquialterates,[17] sesquitertiates, and sesquioctaves; so that Soul, the bond between Mind and Body, was in its essence nothing but Harmony, and made up of harmonies. Undoubtedly they were led to this doctrine by contemplating the fact that the human soul is so greatly delighted by notes which form and contain some harmonious proportions between their magnitudes.

DIGRESSION ON THE PYTHAGOREAN TETRACTYS

From the basic principles set out just above, it appears to be necessary to deduce the Tetractys, the perennial fountain of the human soul by which the Pythagoreans swore. In my opinion that is so because between each pair of the three cubes 1, 8 and 27, for example between 1 and 8, there are two mean proportionals, 2 and 4. Therefore the four numbers 1, 2, 4 and 8, of which the sum is 15, or 1, 3, 9 and 27, of which the sum is 40, make the Tetractys. Now just as pairs of cubes have two proportionals, pairs of squares have one proportional, as is known from the geometers.[18]

Or suppose the Tetractys were 1, 2, 3, 4. 1 is the basis of the numbers. 2 is the first of the numbers and of the evens. 3 is the first of the composites and of the unevens. Moreover by constructing 1 at right angles to 3 a rectangle of area 3 is made, as from an uneven number; but by constructing 2 at right angles to itself, a square of area 4 is made as from an even number, and in the construction of it, it is proper for the length and breadth to be equal, just as in the rectangle on 3 they are unequal. Now the sum of 1, 2, 3 and 4 is 10, and the human soul is accustomed to count in tens. And just as there are four numbers, the same number, that is, as there were Unities in the Fourfold, so also on account of them four kinds of harmonies exist: that between 1 and 2, the Diapason, like that between 2 and 4, and that between 1 and 4, the Disdiapason, which are equivalent to unison; that between 1 and 3, the Diapason Epidiapente, which they held to be the greatest harmony in the system, and is here the second; the third, that between

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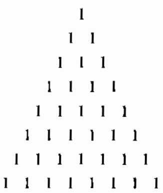
1 1
1 1 1
1 1 1 1

[17] By sesquialterate is meant the ratio of 1 to $1\frac{1}{2}$, by sesquitertiate the ratio of 1 to $1\frac{1}{3}$, and by sesquioctave the ratio of 1 to 1. This terminology was introduced by Nicomachus in his *Introduction to Arithmetic*. There is an English translation by M.L. D'Ooge (1926).

[18] Plato explains in the *Timaeus*, 31C–32A, how two square numbers and two cube numbers can be connected by the insertion of means in continued proportion. In algebraic notation, we have for the squares one mean proportional ab between a^2 and b^2, and for the cubes, two mean proportionals, a^2b and ab^2, between a^3 and b^3.

2 and 3, the Diapente; and the fourth, that between 3 and 4, the Diatessaron. They themselves recognized no further harmonies.[19]

This was in accordance with my own thinking.[20] But on this same Tetractys Joachim Camerarius thinks a little differently, and not a little more correctly, unless his manifold reading of the ancient authors has deceived him. In the Greek commentaries on the golden Poems of Pythagoras[21] he writes as follows.[22]



At first they designated the Tenfold separately by the word Number. Plato was taking it in that sense when he said in the Phaedo[23] that half of a number is always uneven. For let two sets of numbers be defined alternately from Unity to Tenfold. One series will be of unevens, the other of evens, in this way:

1. 3. 5. 7. 9. (Sum 25, which is uneven, and the square of the Fivefold, the number, that is, of the unevens.)

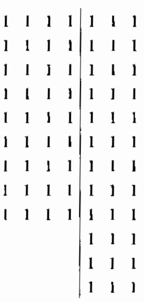
2. 4. 6. 8. 10.

Or with Unity missed out, as the starting point, and the Tenfold as separately called a Number, as follows:

2. 4. 6. 8.

3. 5. 7. 9. (Sum 24, which is even.)

This is the enigma, that the unevens are even. For the separate numbers in the series 3, 5, 7, and 9 are uneven, but taken together they are an even number, four. (And the sum is 24, which is even.)



Therefore the Tenfold, which was called a Number by the Pythagoreans in a special sense, has the property that it is the sum of Unity and its multiples continuously up to the Fourfold. For an equilateral triangle of numbers is constructed, of which the base is the Fourfold, and the vertex, Unity. The Pythagoreans named every number derived from it a Tetractys. For by doubling the sides of the first Tetractys, another Pythagorean Tetractys is constructed, of the number 36, the most famous and in all respects the most useful which they possessed, that is the triangle of numbers of which the base is the Eightfold. Thus they used the number 36 in many demonstrations, especially those concerned with harmony. For in the patterns of the arrangement of this number are found the numbers 12, 9, 8, and 6; and they showed that all the harmonic consonances were contained within these numbers, as represented by the proportions of their intervals. For the number 36 is a square, and its side is 6. It is a triangular

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[19] Ptolemy, in *Harmonica*, Book I, Chapters 6–7, describes the harmonies recognized by the Pythagoreans and criticizes them for rejecting, on numerical grounds, the Diapason epidiatessaron, represented by the ratio 3:8.

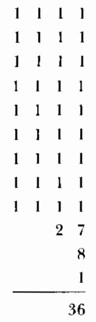
[20] That is, in accordance with Kepler's interpretation of the tetractys.

[21] For a critical edition of the *Golden Poems* of pseudo-Pythagoras, see E. Diehl (1925), vol. 1, pp. 186–194.

[22] Joachim Camerarius, *Libellus scolasticus utilis* (Basel, 1551), 205–208. According to the Pythagoreans, one was the generator of numbers but was not a number itself, so that two was the first female or even number and three was the first male or odd number.

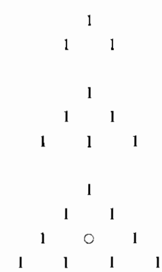
[23] Plato, *Phaedo*, 104A–B.

number, of which the side is 8; it is a rectangle of which one side is 9; and in the other case its side is 12. (For four times 9, and three times 12 both make 36.) Lastly if 6, 8, 9, and 12 are added together into a single sum, the result is the number 35. That is called a harmony by the Pythagoreans, and if Unity is added to it again it completes the number 36. Furthermore of the numbers which have been formed by addition from those which precede in the natural order (that is from the triangular numbers 1, 3, 6, 10, 15, 21, 28) 36 is the first (and the only one below 1225) which is a square, and has as its side 6, the first perfect number (that is to say made up of all its aliquot parts, 3, 2, and 1). The same [36] is also produced by the multiplication of the first two squares, 4 and 9. The same is also formed by addition of, and is made up of, the first two cubes, 8 and 27, together with Unity which is a cube. Because the speculation can be applied in so many ways, this Tetractys was held by the Pythagoreans to be as worthy of consideration and admiration as the foremost; and so they transferred it to Natural Philosophy, and most of all to the contemplation of the soul, and equally to Ethics, and they combined it with some Theology. For as Epiphanius[24] *shows from Irenaeus Against the Valentinians, they made the Tetractys a thing to swear by; but they understood it to mean these four things—Foundation, Silence, Mind and Truth. Though in the golden poems, the formula for swearing is not the Tetractys itself, but he who through the Tetractys showed the permanence of the essence of the Soul. Indeed Plutarch explained the spiritual Tetractys in physical terms, as being Sensation, Opinion, Knowledge, and Mind; and he added the verse,*



> Fountain in which the perennial vein of Nature swells.[25]

Yet the cosmic Tetractys may be more precisely viewed in the following way: from Unity, set out in a threefold way, taking Unity to fill the gap in the middle, and with Quaternaries enclosing it like straight lines, it turns out that this Tetractys produces the tenfold, since on this showing that is the third of the triangular numbers in origin. (For after Unity, the first triangular number is 3, of which the base is 2; the second is 6, of which the base is 3. If you draw three lines enclosing these, through the two points in the former and three in the latter, sketching out a triangle, nothing is left in the middle; but if the third triangular number, 10, with base 4, is given lines enclosing it on the outside, in each case in the positions of its sets of four points, a single point will be left in the middle, which belongs to none of the lines which form the figure but sketches out the space inside, like a heart or kernel.) For this reason the Pythagoreans called the Tenfold



> All-embracing Mother, that encloses all things,
> Unyielding and indomitable and pure,

[24] Epiphanius quotes extensively from the Greek text of Irenaeus, which exists completely only in Latin translation. See J.P. Migne, *Patrologiae cursus completus* (Paris, 1857–1912), vol. 7, col. 447 and vol. 41, col. 491.

[25] Pseudo-Plutarch, *De placitis philosophorum*, Book I.

as Proclus tells us.[26] *And this very completion of ten Units, that is the Tenfold formed by addition from this Tetractys, was reported by the Pythagoreans as containing and accomplishing, or completing, the embellishment of the entire universe; and Plato also follows them. For 1. the universe has become material and sensible; 2. it retains all those things which are in it, indissolubly, by the bond of similarity, or commensurability; 3. it is a whole, inasmuch as it is formed from whole elements; 4. its substance is round in shape; 5. it is that which suffers in itself, and from itself, all that there is to suffer; 6. it moves in a circle; 7. its body is animate; 8. it is the creator of time by means of the revolution of the stars; 9. it indicates certain stars as sacred: they are included in the number of the gods, and make up the Great Year, which is perfect; 10. in every way it is the perfect completeness of things, having in itself all living things, representing four forms (stars representing heaven, birds air, fish water, four-footed creatures earth). On this showing from Unity (as the Pythagoreans say,* "from the cave of the monad"*), there is a progression up to Four (as they say,* until reaching the divine Tetrad itself*), and thus it gives birth to the Tenfold, the mother of all things as we have said. Now the progression of Unity is as follows. One is the world. The Twofold signifies the first multiple contained in it. The Threefold signifies the bond or knot, necessary for the linking together of things; for it is not possible for two single things to combine into one in the absence of the Third.*

The Fourfold is the number which marks out and enumerates the elements. For the world is a solid body, and two solids always require two intermediates, to correspond in continuous proportion. Now their sum (that is, of 1, 2, 3, and 4) is the tenfold, of which we have been speaking all along. For this is the apparel of the completeness, this is its dowry, with which its maker endowed it.

The philosophy of Hermes Trismegistus on numbers.

So quotes Camerarius from the ancients. Most of what Hermes Trismegistus (whoever he was) impressed on his son Tatius agrees with it. His were the words[27]: *Unity embraces the Tenfold on the basis of ratio, and again the tenfold embraces Unity.* Next he makes up the faculty of the soul which is responsible for desire from the twelve avengers, or ethical vices, in accordance with the number of the signs of the zodiac, and makes the body and this power of the soul which is closest to the body subject to it; whereas the same man makes up the rational faculty of the soul from the tenfold ethical virtues. Thus while the Pythagoreans celebrate the Tetractys as the source of souls, and Camerarius says that there is more than one Tetractys, not only that which from the fourfold as base rises to a total of 10, but also above all the other which from the eightfold as base up to its vertex adds up to a total of 36, the said Tatius also hints at the same thing from the teaching

[26] The quotation is from Proclus, In *Platonis Timaeum commentaria*. See the edition by E. Diehl (1903–1906), vol. 1, p. 316 and vol. 3, p. 107. For a French translation, see A.J. Festugière (1966–1968), vol. 2, p. 173 and vol. 4, p. 140.

[27] The writing has the title "Poemandres." F. Patritius, *Nova de universis philosophia* (Venice, 1593).

of his father Hermes when he says it was the time when he himself was still in the Eighth Level, the Eightfold.[28] Indeed the father sent the son back to Pimander singing of the eightfold. There in fact occur the eightfold ethical casts of soul, seven corresponding with the seven planets, as is apparent, starting from the Moon; but the eighth, more divine and more at rest, to the idea, I think, of the sphere of the fixed stars. Furthermore everything is carried out through harmonies. There is much impressing of *silence*, much mention of *mind* and *truth*. Also the cave, the foundation, the inner sanctum, the mixing bowl of spirits, and many other things are evinced, so that there can be no doubt that either Pythagoras is playing Hermes or Hermes Pythagoras.[29] For there is the additional fact that Hermes expounds a particular theology, or cult of a divine power. Often he paraphrases Moses, often the Evangelist John in his sentiments, especially on regeneration. He impresses on his disciple certain ceremonies; whereas the authorities declare the same of the Pythagoreans, that part of them were given over to theology and to various ceremonies and superstitions, and Proclus the Pythagorean locates his theology in the contemplation of numbers.

John 3.

So much by way of digression. Let us now return to the Pythagorean demonstration of the harmonious proportions.

The error of the Pythagoreans about the number of harmonies.

For the Pythagoreans[30] were so much given over to this form of philosophizing through numbers that they did not even stand by the judgment of their ears, though it was by their evidence that they had originally gained entry to philosophy; but they marked out what was melodic and what was unmelodic, what was consonant and what was dissonant, from their numbers alone, doing violence to the natural prompting of hearing. This harmonic tyranny of theirs lasted up until Ptolemy, who was the first, one thousand five hundred years ago, to uphold the sense of hearing against the Pythagorean philosophy, and accepted as melodic not only the proportions stated above, and the proportion of one and an eighth to one as equivalent to a Tone, but also admitted the proportion of one and a ninth to one as equivalent to a minor tone, and that of one and a fifteenth to one as equivalent

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[28] By relating the eighth sphere to which the soul ascends, as described in the *Corpus Hermeticum*, with the Pythagorean four as the number of the soul, Kepler seems to endorse the conformity of the Hermetic teachings with the Pythagorean harmony. See *Corpus Hermeticum*, edited by A.D. Nock (1945 and 1954), vol. 2, pp. 200–209. See also Frances A. Yates (1964), pp. 441–442.

[29] Kepler seems to be uncertain whether Hermes was influenced by Pythagoras or vice versa. Isaac Casaubon in 1614 established that the Hermetic writings were of post-Christian origin.

[30] The view that Kepler here attributes to the Pythagoreans was held even more strongly by Plato and his followers. For Plato criticized the Pythagoreans for "preferring their ears to their minds," Plato, *Republic*, 530D–531A. See also Barker (1984), p. 244. Kepler goes on to criticize more of Plato's ideas that he attributes to the Pythagoreans.

to a semitone.[31] He did not only add other proportions of one and a single aliquot part of one, which were sanctioned by the ears, such as one to one and a quarter or one to one and a fifth, but he also added some of the proportions of several aliquot parts, such as the proportion of 3 to 5 and 5 to 8 and others.

Ptolemy's error about the number of harmonies and melodic intervals.

On this showing Ptolemy did indeed correct the Pythagorean speculation on the origin of the harmonic proportions as forced, but did not completely eliminate it as false; and the man who restored the judgement of the ears to its rightful place in words and doctrine nevertheless deserted it again, as even he adhered to the contemplation of abstract numbers. For the cause of the number of the harmonic proportions and of the individual proportions is not, even so, adequate for its effect; but in designating the consonances it falls short, in the case of the other melodic intervals it goes too far. Ptolemy still denies that the thirds and sixths, minor and major (which are covered by the proportions 4:5, 5:6, 3:5 and 5:8) are consonances, which all musicians of today who have good ears say they are. On the other hand he accepts the proportions 6:7, 7:8 and others among the melodic musical intervals, so that if a tune proceeds from UT to FA, a note is constituted, intermediate between RE and MI, in the proportion in which 7 is the middle term between 6 and 8. Let this note be RI, so that we can refer to it. Then it is possible to sound[32] UT, RI, FA, just as it is possible to sound UT, RE, MI, FA, which is utterly abhorrent to the ears of all men and the usages of singing, even though it may be possible for strings to be tuned in that way, seeing that as they are inanimate they do not interpose their own judgement but follow the hand of the foolish theorist without the least resistance.

His error in treating a non-cause as a cause.

Furthermore if both the cause which was sought in abstract numbers, and the effect, that of consonance, were as far as possible equal in scope, and it could without absurdity be seen as the archetypal cause, bearing witness that it was from the contemplation of those numbers that the Father of things, the Eternal Mind, took the idea of notes and intervals, and so that they should be pleasing to human spirits He had to arrange them in the shape of those spirits, yet it would still not be very clear why the numbers 1, 2, 3, 4, 5, 6, etc., conform with musical intervals, but 7, 11, 13, and the like do not conform. Also, the cause of this fact would not be revealed by the numbers, as numbers, from

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[31] These three melodic intervals make up the fourth or tetrachord in Ptolemy's scale called "diatonon syntonon" (tense diatonic), *Harmonica*, Book II, Chapter 1. This scale, with the addition of an extra note introduced by Guido d'Arezzo, has just consonances, except for a minor third, fifth and minor sixth narrowed by a comma on the second degree. Cf. Kepler's system in Chapters 11 and 12. Zarlino called Ptolemy's scale "sintono artificiale" and claimed that singers always used just intonation, a scale that he called "sintono naturale." *Sopplimenti musicali* (Venice, 1588), 140–149. Such a scale, however, is very unstable. Cf. D.P. Walker (1978), 15–17.

[32] This is Ptolemy's "chroma syntonon." Harmonica, Book II, Chapter 1.

within themselves. For the cause drawn from the Threefold basic principles, and the family of squares and cubes derived from them, is no cause, since the Fivefold is foreign to it, although it refuses to have its rights of citizenship in the origin of musical intervals torn from it.

Yet not even this is satisfactory to the theorist, for he knows that the numbers 1, 2, 3 are symbols of the basic principles of which natural things consist. For an interval is not a natural thing, but a geometrical one. Hence unless these numbers number something else, which is more akin to the intervals, the philosopher will not be able to put any confidence in this cause but will suspect it of not being a cause.

For these reasons,[33] then, for the last twenty years in order to work this out fully I have set myself the task of illuminating this part of Mathematics and Physics, by discovering causes which on the one hand would satisfy the judgement of the ears, in establishing the number of the consonances, and the other melodic intervals, without trespassing beyond what the ears bear, but which on the other hand would set up a clear and overt criterion between the numbers which form musical intervals and those which have nothing to do with the matter, and lastly which, with respect both to the archetype and to the Mind which uses the archetype to shape things to fit it, would have a kinship with the intervals, and so would rest on the clearest probability. For since the terms of the consonant intervals are continuous quantities, the causes which set them apart from the discords must also be sought among the family of continuous quantities, not among abstract numbers,[34] that is in discrete quantity; and since it is Mind which shaped human intellects in such a way that they would delight in such an interval (which is the true definition of consonance and discordance) the differences between one and the other, and the causes of such intervals' being harmonious, should also have a mental and intellectual essence, that is that the terms of the consonant intervals are properly knowable, but those of the dissonant intervals either cannot be properly known or are unknowable. For if they are knowable, then they can enter the Mind and into the shaping of the archetype; but if they are unknowable (in the sense which has been explained in Book I) then they have remained outside the Mind of the eternal Craftsman, and have in no way matched the archetype. But more will be said on these points when we describe the actual theory Chapter by Chapter;

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[33] One of Kepler's fundamental principles, stated in a letter to David Fabricius in 1603 (KGW 14, p. 412), is that hypotheses must be built upon and confirmed by observations.

[34] Here Kepler emphasizes that his harmony is geometrical and not arithmetical, like that of the Pythagoreans. Again the geometrical harmonies are perceived by the mind. As early as 1597, Kepler remarked to his teacher Michael Maestlin that, as the eye is for colors and the ear for sounds, so is the mind or intellect for knowledge of quantity (KGW 13, p. 113). Cf. Plato, *Timaeus*, 46D–47E. See also Field (1984a).

and may we embark on it with God's help. Throughout we shall indeed speak of melody, that is harmonious intervals which are not abstract but realized in sound; yet to the educated ears of the mind the underlying reference throughout will be to the intervals abstracted from the sounds. For it is not only in sounds and in human melody that they yield their charm, but also in other things which are soundless, as we shall hear in the fourth and fifth Books.

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CHAPTER I.

On the Causes of Consonances.

Definition

Although the ancients used the following terms, "monophonic, antiphonic, homophonic, diaphonic, symphonic, asymphonic," we shall use "dissonant" to mean the same as "asymphonic," and "consonant" to mean the same as "symphonic." This last is differentiated into "identical," which we shall adopt instead of "homophonic," and "non-identical," which we shall adopt instead of "diaphonic."[35] And of the identical there are two kinds, single-sounding, and identical by opposition.[36]

Definition

In geometry the terms "part" and "parts" are different. For the term "part" is used for that of which the whole is a multiple in a certain proportion, such as double, triple, quadruple; whereas "parts" is used when not a single and unique whole but a quantity of wholes is a multiple of them. Thus one seventh is called a part because the whole circle is seven times the part; but three sevenths are called not a part but parts, because a total of three circles is seven times the arc.

Here however we shall not use that distinction; but we shall call a part one of the fractions mentioned as much as the other. That is, every fraction which is expressible[37] we shall call a part, though with the restriction, provided it is not greater than a semicircle.

The term "remainder" on the other hand will be used for what is left, being not less than a semicircle, when an expressible portion [in length] is subtracted from the whole.

The distinction between a remainder and a part is extremely necessary, because a part can be a consonance, and its remainder a dissonance, as we shall see.

Definition

A string [chord] is here taken to mean not the line subtended by an arc of a circle, as in geometry, but any length which is capable of emitting

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[35] Kepler gives these terms in Greek. By identifying the non-identical concords with the Greek term Διάφωνον, Kepler departed from the ancient and medieval usage, according to which the term meant dissonance.

[36] Single-sounding means in unison and identical by opposition means separated by one or more octaves. See Ptolemy, *Harmonica*, Book I, Chapter 7.

[37] See *Harmonice mundi*, Book I, definitions XII and XIII.

a sound; and as a sound is elicited by motion, "string" is to be understood in the abstract in reference to the length of any motion whatever, or to any other length whatever, even if it is conceived in the mind.

Axiom I

The diameter of a circle, and the sides of the fundamental figures expounded in Book I, which have a proper construction, mark off a part of the circle which is consonant with the whole circle.

How the circle can be stretched out, so that it emits sounds, and how it must be fastened to a hollow body, so that resonance occurs, either at one mark so that the whole sounds, or at two, so that the parts sound, it would be a lengthy business to expound here. However it was necessary to start in this way because it is not only a question of melody, which is harmony realized in sounds, but the underlying reference to an interval in abstraction from sound must be understood. As far as music is concerned, it is sufficient that a string stretched out straight can be divided in the same way as when it is bent round into a circle it is divided by the side of the inscribed figure.

Corollary

The consonances are infinite, because the constructible figures are infinite.

However it is not yet time to speak of the identification of consonances, which does not make itself very obvious. On this point the Pythagoreans sought in their numbers, as causes, the bounds of the size of consonant intervals, which only the human hearing fixes for them, which is not of infinite power. The restriction of the number of consonances by the abstract harmonic intervals is therefore only accidental, and not causal. Even the musicians of today themselves overstep the Pythagorean bounds, to say nothing here about celestial harmonies.

Axiom II

To the same extent as the construction of a side is remote from the first degree, the consonance of a part of a circle, cut off by the side, with the whole circle, deviates from the most perfect consonance of unison; or, the allotted place of the figure of which it is the side among other figures is the same as the place of that consonance among the others.

This subordinate axiom will be adopted for the identification of consonances, with respect to their giving pleasure.

Axiom III

The sides of the regular and star figures which are not constructible mark off a part of the circle which is dissonant from the whole circle. The same applies to the side of a figure which is in fact constructible but not in its own right, nor by a proper construction. Or in place of the lack of a proper construction, consider lack of congruence, as in Book II. By both methods the fifteen sided figure is excluded.

This axiom will round off and complete the cause of consonance which I am substituting for the Pythagorean abstract numbers which have been repudiated.

Corollary

Then these parts are dissonant	from the whole
1, 2, 3	7
1, 2, – 4	9
1, 2, 3, 4, 5	11
1, 2, 3, 4, 5, 6	13
1, – 3, – 5, – –	14
1, 2, – 4, – – 7	15
1, 2, 3, 4, 5, 6, 7, 8	17[38]
1, – – – 5, – 7, –	18
1, 2, 3, 4, 5, 6, 7, 8, 9	19 *and so on to infinity.*

Axiom IV

Figures which have kindred constructions for their sides, also give rise to kindred harmonies.

Through this axiom the origin and cause of the harmonic proportions will be proved superabundantly.

Axiom V

Strings or arcs of a circle, of equal tensions, having to each other, with respect to their length, the same proportion as the Part or residue of a circle has to the whole circle, also have the same consonance or dissonance, although it occurs between different limits or sounds.

Let it thus be understood in the abstract that a circle stands in certain harmonic proportions to its part; and that within whatever various limits, whether sounds, or soundless motions they are found, they are always harmonic.

Now this axiom is added, because not all harmonic proportions arise immediately from the circle itself, directly from its division by means of a regular figure, but some accrue which are generated from the prior ones themselves, up to a certain limit, as we shall see in the propositions.

The application of this axiom is in Propositions VII and VIII.

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[38] Although Kepler believed the 17-sided polygon to be excluded on both counts, it is in fact only excluded on the count of congruence. This polygon was first shown to be constructible by Gauss in the nineteenth century. Note that the parts 2, 4, 6 are excluded as parts which are dissonant with the whole 14, because they have already appeared in the heptagon. Also the parts 2, 4, 8 in relation to the whole 18 have already appeared in the nonagon.

Axiom VI

When two strings emit identical sounds, a third note which is consonant with one of them will also be consonant with the other; and one which is dissonant from one will also be dissonant from the other, and so with various different kinds of consonances or discords also.

Note that identity of sound is put in the subordinate position as a species, and consonance in the antecedent position as genus. Hence the two following points should be understood. First, it does not follow that if two strings are consonant in any way, then a third is also consonant with both of them, or dissonant with both of them. For that is false of the genus, but true of the species of identity of sound. Secondly, it does not follow that if the third is consonant with one of the sounds which are identical in some particular consonance, it will be consonant with the other with the same species of consonance; for that would not always be true, which I shall demonstrate with an example, though in anticipation. Let there be two notes, making the diapason G and g. Let there be a third note d. It makes a fifth with G, and is therefore also consonant with g; yet not by a fifth, but by a fourth.

The chief application of this axiom is in Proposition IV.

Axiom VII

When two strings or voices emit identical sounds, a third note which is identical in sound with one of them, will also be identical in sound with the other.

What could not be affirmed in the previous axiom of the genus, is true in the species as to identity of sound.

The application is in Proposition III.

The cause of the harmonies metaphysical.

Then contemplation of these axioms, especially of the first five, is lofty, Platonic, and analogous to the Christian faith, looking towards metaphysics and the theory of the soul. For geometry, the part of which that looks in this direction was embraced in the two previous books, is coeternal with God, and by shining forth in the divine mind supplied patterns to God, as was said in the preamble to this Book, for the furnishing of the world, so that it should become best and most beautiful and above all most like to the Creator. Indeed all spirits, souls, and minds are images of God the Creator if they have been put in command each of their own bodies, to govern, move, increase, preserve, and also particularly to propagate them.

Then since they have embraced a certain pattern of the creation in their functions, they also observe the same laws along with the Creator in their operations, having derived them from geometry. Also they rejoice in the same proportions which God used, wherever they have found them, whether by bare contemplation, whether by the interposition of the senses, in things which are subject to sensation, whether even without reflection by the mind, by an instinct which is concealed and was created with them, or whether God Himself has expressed these proportions in bodies and in motions invariably, or whether by some geometrical necessity of infinitely divisible material, and of motions through a quantity of material, among an infinity of proportions which are not harmonic, those harmonic proportions also occur at their own time, and thus subsist not in BEING but in BECOMING. Nor do minds, the images of God, merely rejoice in these proportions; but they also use the very same as laws for performing their functions and for expressing the same proportions in the motions of their bodies, where they may. The following Books will offer two splendid examples. One is that of God the Creator Himself, who assigned the motions of the heavens in harmonic proportions. The second is that of the soul which we generally call Sublunary Nature, which actuates objects in the atmosphere in accordance with the rules of the proportions which occur in the radiations of the stars.[39] So let the third example, and the one which is proper to this Book, be that of the human soul, and indeed also that of animals to a certain extent. For they take joy in the harmonic proportions in musical notes which they perceive, and grieve at those which are not harmonic. From these feelings of the soul the former (the harmonic) are entitled consonances, and the latter (those which are not harmonic) discords. But if we also take into account another harmonic proportion, that of notes and sounds which are long or short, in respect of time, then they move their bodies in dancing, their tongues in speaking, in accordance with the same laws. Workmen adjust the blows of their hammers to it, soldiers their pace. Everything is lively while the harmonies persist, and drowsy when they are disrupted.

Whether these and the like are intentional or involuntary, that is the work of the mind; and whether it is by the necessity of the nature of the elements and of matter that no tuning can suit the senses but that which is based on the harmonic proportions of the figures, has been argued in various ways by the philosophers. All ask the source of that pleasure which glides into the ears from the proportion of notes, pleasure by which we define consonances. Those who incline towards matter and the motion of the elements, adduce as an example the fact, in itself indeed certainly remarkable, that a string which is set in motion draws another string which has been set in motion into sounding with it, if it has been tightened into consonance with itself, but if it has been tightened into dissonance leaves it motionless. Since that cannot come about by the intervention of any mind, because the sound, the supposed cause of it, does not have mind or understanding, it follows that we can say it comes about by the adjustment of the motions to each other. For the sound of the string has higher or lower pitch, from the speed or slowness of the vibration with which the whole

Remarkable discovery on strings.

[39] What is involved here is the influence of the aspects on the weather.

free length of the string vibrates.[40] These differences in the sounds do not arise primarily and immediately in the actual length or shortness, but secondarily, that is to say because when the length is diminished the slowness of the vibration is diminished, and its speed is increased. The reason is that, if the free length of the string remains the same, the actual tightening of it raises the pitch of the sound, because by leaving the string less slack, it also diminishes the space through which it can vibrate in its reciprocating motion.[41]

Then if the tension of two strings is equal, so that they can sound in unison, in that case the sound of one, that is the immaterial emanation of the body of the string,[42] which is set in vibration, gliding from its string, strikes the other string, just as when someone shouts at a lute, or something else hollow. With that shout he strikes the hollow object and makes all its strings resonate. Now that emanation of the vibration strikes the other string with the same rhythm of speed and the latter also moves in that rhythm because it is equally tight; so that individual beats (into which the vibration is understood to be divided) continually come upon individual stationary points[43] of the other string as they strike it. So it comes about that the string which is tightened to unison with the first moves most of all. Yet the string which is of twice or half the speed also moves, because two beats of the vibration are completed for one stationary point of the string, and thus every third beat after the previous one always coincides with the extreme of one stationary point. Lastly the string which is of one and a half times the speed also moves to some extent, because three little beats occur for two stationary points of the former string. But now the beats on the one hand and the stationary points on the other begin to meet each other more frequently and to impede each other. While two beats of the former string miss the end of a stationary point of the latter, only one coincides; and when they meet in that way the motion of the other strings is halted, exactly as if someone had applied a finger to the one which was vibrating. This seems to me the remarkable cause of this discovery; and if anyone is more fortunate than I in his intellectual search, I shall yield him the palm.

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[40] The phenomenon of resonance had been observed since the time of Aristotle but was first accurately described in 1677 by students of John Wallis, namely William Noble and Thomas Pigot, who showed experimentally that only the overtones in unison are incited. *Philosophical Transactions of the Royal Society of London*, 12 (1677), 839–842.

[41] While Kepler is correct in believing that the pitch depends on the frequency of vibration and thus in turn on the tension of the string, the reason he gives is false. The amplitude of the vibration determines only the loudness and not the pitch. Moreover the amplitude is not restricted by the tension.

[42] Kepler evidently supposes sound to be an immaterial emanation similar to light and the force which moves the planets.

[43] These are not the stationary points now called nodes. Kepler is referring to the extreme positions of displacement of a string, where the string is instantaneously stationary before moving back again towards the mean position.

What follows, then? If the speed of one string has the power to move another which is in proportion to it, but which, as far as can be seen, remains untouched, will not the fact that two strings have the same speeds as each other have the power to titillate the hearing pleasantly, on account of the fact that in a way it is moved uniformly by both strings, and that two beats from two sounds or vibrations co-operate in the same impulsion? It is vain, say I, to dispose of this matter so easily; and I wonder that Porphyry[14] the commentator on the *Harmonics* of Ptolemy could have been satisfied with something like that for the cause of this phenomenon, although he is a philosopher of the most profound insight. Unless, as is probable, he was constrained by the difficulty of seeking out the cause from penetrating as far as he wished, and thought it better to make some statement than to be completely silent, which they always say is a disgrace to a philosopher. For what, I ask, is the proportion of titillation of the hearing, a corporeal thing, to that unbelievable pleasure, which we feel totally within the mind from harmonic consonances? Surely if any pleasure does come from the titillation, the chief participant in that pleasure is the organ which undergoes the titillation? For it seemed to me that every sense should be defined in this way, in the *Dioptrics*,[15] because the particular sensation is complete, generating pleasure or pain, when the emanation of the organ which is ordained for that sensation, as it is affected by the external circumstance, comes within the tribunal of the common sense, by the passage of the spirits. Yet in fact in the hearing of consonant notes or sounds, what parts of the pleasure, I ask, are attached to the ears? Surely we are pained sometimes by our ears, when we gape at what we hear, and put a hand in the way of excessive noises; yet we are no less eager to perceive consonances, and our hearts leap within us? Add the fact that this explanation deduced from the motion applies particularly to unison, whereas it is not unison which is especially pleasurable, but other consonances, and their combination. Much can be adduced to overthrow this explanation which has been adduced for the pleasure of consonances, which I refrain for the present from setting out in too much detail. I emphasize a single

Of what kind is the cause of the pleasure in harmony?

What is a sense?

The cause of harmonies' being pleasurable to be sought in the approbation of the mind.

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[14] Porphyry's commentary had not appeared in print. Kepler used a manuscript copy he had borrowed from Herwart von Hohenburg. What Kepler here attributes to Porphyry is an explanation of consonance in terms of a coincidence of vibrations. As he goes on to point out, if coincidence of vibrations brought about the sweetness of sound, then the unison should be the sweetest, since in this case the coincidence is most complete, whereas the ear prefers other combinations. Kepler therefore rejects this explanation of consonance, preferring his own emanation theory. With Galileo Galilei and Marin Mersenne, however, the coincidence theory of consonance became a starting point for an experimentally based science of acoustics. See H.F. Cohen (1984), pp. 90–105.

[15] See J. Kepler, *Dioptrice*, proposition 61 (KGW 4, pp. 372–373). The common sense is an internal faculty of the sensitive soul which forms judgments concerning the operations of the particular senses. See Aristotle, *De anima*, 424 b 20–427 a 18.

point, which I have already touched on above, and which can represent the whole: that the operations and motions of bodies, which imitate the harmonic proportions, are on the side of the soul and the mind, assigning them a cause for their delight in consonance. Nor is the authority of the ancients against them. When they defined soul now as motion, now as harmony, it was not so much that they spoke absurdly as that they were interpreted inappropriately, since in difficult matters there often lurk mystical senses concealed beneath the husks of the words. Indeed the philosophy of Timaeus the Locrian[16] on the composition of the soul from harmonic proportions, mentioned in the preamble, was refuted by Aristotle[17] in the sense conveyed by the actual words; but I should not dare to affirm that there is nothing lurking in those writings but what the actual words convey. On the contrary I think no-one will deny that the author at least holds what I here ascribe to him, that it is Mind or the human intellect by the judgement or instinct of which the sense of hearing discriminates pleasant, that is consonant proportions from the unpleasant and dissonant, especially if he ponders carefully that proportions are entities of Reason, perceptible by reason alone, not by sense, and that to distinguish proportions, as form, from that which is proportioned, as matter, is the work of Mind.

From the knowledge-producing construction of the figures.

Now since we have expounded two properties of the regular figures,[18] the knowledge-producing constructibility of the sides in each case, and the congruence of those which are wholly linked to each other, which clearly do not both apply over the same range, our axioms refer chiefly to constructions, because that is more closely associated with the proportions of the motions, from which sounds are also derived.

For congruence belongs to figures as wholes; whereas motions (in which harmonic proportions occur) extend in a straight line the sides of the figure from which they are derived (since generally all of them are considered as rectilinear) and thus undo and destroy their own figure, as serpents do their mother. A figure, insofar as it is congruent, divides a complete circle into parts: the harmonic proportions extend the divided circle into a straight line, and cancel the effect of the division made by the figure. Thus consonances along with constructible figures reach to infinity: congruent figures are limited by the twelvefold number. Lastly any figure makes a single division of a circle; but the parts established in a circle always make two consonances with the whole.

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[16] Plato, *Timaeus*, 35A–36E and 47D.

[17] Aristotle, *De anima*, 407 b 26–408 b 18.

[18] Here Kepler states his intention to base his musical theory primarily on the property of constructibility of the regular polygons, described in Book I, though the property of congruence, described in Book II, will also play a minor role.

Although in fact the argument in this third Book will be more concerned with the knowledge-producing construction of sides than with the congruence of complete figures, nevertheless on account of their close relationship the latter will not be neglected in its place. For first the Latin meaning of the word congruence, if you make a thorough investigation, is the same as that of the Greek word harmony—the words with which we shall deal in this Book—except that usage has quickly made a distinction between these words from the subjects to which they refer. Secondly, the congruence of figures imparts a certain congruence to motions (with which this and the fifth Book will be concerned). Thirdly, although we are examining not so much the whole figure as one side of it, and it is the part which that side cuts off which is consonant, yet at the same time it is also true that we are not so much considering the size of the part of the circle which is intercepted as the nature of the figure by which that is done, whether it is constructible and congruent, or the contrary. For any figure has, from its angles through which it was allotted congruence in Book II, also acquired a construction in Book I. The examination of the congruence of figures is therefore not to be dissociated from harmonies.

And from congruence.

Proposition I

The consonance of a half with the whole, apart from unison, is the only one which is in the first degree simple, perfect and identical, that is identical by opposition.[49]

For that which is in the nature of a figure is made up of diverse elements, and is therefore not simple or identical. For a figure has area, and parts in respect of its area, and angles which differ in position. On the other hand that which is not in the nature of a figure, because of course it is without breadth of area, and in that respect also without parts, and angles, being merely a straight line, and of a measure equal to that proposed, is therefore itself both simple, and the same in its measure, that is identical. The regular figures are in fact of the former kind, when they are inscribed in a circle; the diameter of a circle is of the latter kind. For 1. all the sides of figures diverge equally from the center: the diameter passes through the center itself. 2. A chord,[50] which divides a circle from a point with the side of a figure as measure, when it has proceeded to do that a number of times, eventually returns with the other end of the side to the first point: the diameter on the other hand, passing through the center itself, returns at once at the first repetition to the initial point. 3. The rest of the figures possess both length of sides and area of the surface which they surround: the diameter, which neither surrounds nor encloses any part of a plane, in continual

[49] In this proposition, Kepler seeks to establish by adducing many reasons the primacy of unison and after that, of the consonance which he later called the octave.

[50] Kepler seems to mean that, starting at any point of the circle and taking the side of the figure as a measure, the starting point will again be reached after a whole number of divisions.

repetitions wholly coincides with itself every second time. 4. The other figures when they divide a circle make many parts: the diameter makes the smallest of all numbers of parts, that is two; for if it is to divide the whole, it absolutely could not make fewer parts than two. 5. And since the diameter is the measure with which the sides of the figure are to be compared, for purposes both of conception and of construction, the sides of the remaining figures are more laborious to draw, and are brought to achievement of knowledge with a more imperfect degree of construction; but the diameter of a circle is drawn according to the simplest law, so that it passes through the center, from one point on the circle to the one opposite, and is equal to itself, and the measure of itself. 6. Also the sides of the figures in a single division of the circle, or in the cutting off of a part, make unequal portions, and a part which is smaller than the remainder: the diameter leaves a part cut off which is equal to the remainder. Now this proportion of equality is pure and simple and perfect, because parts which are equal among themselves, are as far as mensuration is concerned the same thing. 7. Lastly, the other figures do indeed divide the circumference of the circle into a number of equal parts, but the area of the circle into a number of unequal parts, because one—that is, the area of the figure—is left in the middle which is larger than any one of the segments: the diameter divides not only the circumference but at the same time the area also into two equal parts.

But by Axiom II the character of the side or line which divides the circle consonantly passes over to the consonance itself. Therefore the consonance of the part which the diameter cuts off from the circle, that is of the semicircle with the whole circle, is simple, perfect, and identical. Also by Axiom V all other lengths which are to each other as the whole circle is to half of itself, make the same, that is identical, perfect, and simple consonance. Further, in the case of numbers (not certainly of abstract and counting numbers, but of lengths which are counted numbers)[51] *the double proportion, that is between 1 and 2 and also between equal multiples of them, gives rise to identical consonance.*

Note here how the diameter through all its simplicities and perfections is nevertheless not as simple as a point, but remains a line bounded by two points of the circle, cutting the circle in opposite positions, and establishing two parts. Just as those parts, although they are equal to each other, are individually less than their own whole, so also an identical consonance is nevertheless not a unison; and of notes although they are in identical consonance yet one is smaller, the other larger. That is, the former is high, the latter low, corresponding with the former, so to speak, from the opposite side; so this is called an identical consonance by opposition.

[51] The distinction between *numeri numerantes* (counting numbers) and *numeri numerati* (counted numbers) is explained in the appendix to Book V of the *Harmonice mundi*. The former are abstract numbers (whose properties are accidental), the latter concrete numbers or numbers embodied in real things: that is, for Kepler, numbers embodied in geometrical objects such as regular polygons and the Platonic and Archimedean solids. The distinction of the kinds of number was made by Aristotle, *Physics*, 219b3–9.

You have, then, from the diameter of the circle the true cause through which the sound of a whole string with the sound of half the string, though they are different from each other, is yet taken by the hearing as in a way the same in comparison with the other consonances.

Others seek for the cause of this identity of sound in the number of the eight notes, vainly, since this identity of sound is by nature prior to the division of this interval into the seven melodic parts by which the eight sounds are designated.

However it is not yet time to give a name to that consonance, nor to the rest; for that must be deferred to Chapter V.

Yet notice also the fact that other parts also are identical consonances although they are not established by the diameter, but not in the first degree, nor through the figures, but through their propagation, which is the subject of the following propositions.

Proposition II

If of two parts of a circle the smaller is to the larger as the larger is to the whole circle, in some other proportion than successive doubling, then if the larger is in consonance with the whole circle, the smaller part will be in dissonance with it.

For after the double comes the triple. Now successive tripling puts in third place the ninth part of the whole circle, successive multiplication by five the twenty-fifth part; and successive multiplication by six implies the ninth part, successive multiplication by ten the twenty-fifth part, because six times six is 36 which is four times nine, and ten times ten is 100, which is four times 25. And likewise for the rest. But a ninth, and a twenty-fifth, and similar parts are dissonant from the whole, by Axiom III. See Proposition XLVII in the first Book.

Proposition III

Strings in the proportion of successive doubling are in identical consonance with each other, but those in more distant proportion are in consonance at a more remote degree.

For the three nearest are to each other as the whole circle is to the half, and to the quarter respectively. But both the half and the quarter are in consonance with the whole circle, by Axiom I. Also the quarter is in consonance with the half, by Axiom V. Therefore all the three nearest proportions are in consonance with each other. Further, the consonance of the quarter with the whole circle is also identical. For a whole and its half are in identical consonance, by Proposition I. So also is the quarter with the half, by the same Proposition: hence by Axiom VII the quarter is also in identical consonance with the whole circle; and by Axiom V any fourfold is with the single.

Now indeed the ratio which is between the first, second, and third proportionals will be the same as that between the second, third, and fourth, and so on continuously between the three which are nearest to each other. Therefore *all proportionals which are in the proportion of successive doubling are in identical consonance with each other.*

Notice therefore in such cases the distinction between consonance as a genus and identical consonance as a species. Fourth, eighth, sixteenth, and similar parts also are in consonance by Axiom I and the figures, Tetragon, Octagon, and so on: but they are in identical consonance, on account of the progressive generation of this class of figures from the bisection of the circle.

For if it had not had this derivation their consonances would not have been identical. For as all figures make either many parts of the circle, if they are equal, or unequal parts, if they make only two of them, since they enclose an area, they do not divide the area of the circle equally, nor do their sides pass through its center, nor do they return to the same point, nor are they equal to its diameter: furthermore consonances derived from figures of the tetragonal class would in some way have amplified themselves to the hearing, and stretched the mind by the manifest variety and diversity of their notes, as do the consonances which come from the other figures, which consist of a number of sides which is not a product of successive doubling, by Proposition I.

However not all power has been removed from this class of figures of varying the consonances and diverting them from the purity of identical consonance (just as they themselves have regressed from the simplicity of the diameter). For first, although the consonance of the part of the circle cut off by the figure is converted into pure identical consonance (on account of the said derivation of the parts of the circle, from the original bisection), yet the degrees of identical consonance become more remote, for the smaller which is in identical consonance by opposition with the one next larger than itself becomes continually higher in pitch as the points of opposition are multiplied. Thus the intervals of the notes continually increase. Secondly, identical consonance does indeed remain in the part (as in division by the diameter), but not at all in the remainder; for this remainder in the later figures becomes continually inferior as far as its harmonic nature is concerned. But there follow particular propositions about such remainders.

And on the other hand, it is not only the tetragonal class which generates identical consonances; but also the other classes, to the same extent as they partake of bisection, also make identical consonances. For the part of a circle cut off by the side of a subsequent figure is always in identical consonance with the part cut off by the side of an antecedent figure, as the remaining propositions relate. Thus the analogy holds good in all its branches.

The application of this proposition is in the following one.

Proposition IV

A string which is in consonance with either one of two multiples in the proportion of continuous doubling is also in consonance with the remaining one; and if it is in dissonance with one, it is also in dissonance with the other.

For by Proposition III sounds which are in the proportion of continuous *doubling are identical with one another. However, what is in consonance with one of two identical strings is also in consonance with the other; and the rest follows, by Axiom VI.*

Axiom VI was assumed for the sake of this proposition; and this proposition is now of service in examining the parts and remainders of circles. Let know-alls beware of abridging the Propositions and Axioms; for there is no tautology: everything is necessary. Anyone who wants to get through the matter too quickly will get himself into a tangle.

Proposition V

Although the additional sides of stars are constructible, on account of their constructibility they determine the consonant parts of the whole in a circle on the same footing as their fundamental figures do, as in Axiom I; however those which cut off a part of a circle which consists of the appropriate number (of the parts which the fundamental figure made) for some inconstructible figures are excepted, when the numbers of the part and of the whole have no common factors.

The first part of this proposition is an axiom. So that it should not be made too general, it had to be restricted by the second part of the proposition. Now the proof is as follows. For let there be a circle divided by a constructible figure, for example by an icosigon. Now let there be an icosigonal star, the side of which subtends nine of the twentieths made by the icosigon, in the proportion of 9 and 20 which have no common factors. Then since the part has been cut off from the circle, it will certainly be smaller than the whole. Yet it may be larger than half of the whole, or a quarter, or an eighth, and so on by dividing it repeatedly until some part of the whole in the ratio of continuous halving is less than half of the part with which we are concerned. Thus in our example if the whole is 20, the part with which we are concerned is 9. Take half of the whole, 10, and half of that again, 5, and a third time, $2\frac{1}{2}$, an eighth of the whole. That is now smaller than half of nine. Then our part, 9, is to an eighth of the whole circle, $2\frac{1}{2}$, as a circle divided by an unconstructible figure to some part produced by its own division, that is as 18 to 5. Now the Corollary of Axiom III declared that five eighteenths are in dissonance with the whole 18. Therefore by Axiom V the part produced by our division, 9, will be in dissonance with an eighth of the circle ($2\frac{1}{2}$ parts by our division). Therefore by Proposition IV our part, 9, will also be in dissonance with the whole circle, 20, although its chord is constructible, but at the most remote degree; and its star is among the incongruent figures.

Proposition VI

The remainders of circles or strings, after parts in consonance with the whole have been cut off, if they are in the proportion of continuous doubling with their consonant part, are in consonance both with the part cut off and with the whole circle or string.

With the part cut off by Proposition I, with the whole by Proposition IV.

Proposition VII

If such a remainder is in the same proportion to the half or quarter of a circle or string as the whole circle is to some other part of itself which is consonant, it will also be in consonance with the whole circle; if it is in the same proportion as a dissonant part, it will be in dissonance.

For the whole circle, and its half, and its quarter, are in the proportion of continuous doubling. Hence (by Proposition IV) those remainders which are consonant with such a part of the circle are also consonant with the whole; and those which are in dissonance with the former will also be in dissonance with the latter. But those remainders are consonant with such a part which are in the same proportion to it as the whole to any consonant part; and those remainders are dissonant with such a part which are in the same proportion to it as the circle to any dissonant part. That is by Axiom V.

Therefore such remainders are also in consonance with the whole circle; and those of the opposite kind are in dissonance with the whole circle.

This Proposition is for the sake of the following Proposition VIII.

Proposition VIII

However if a remainder is in the same proportion to a cut off part as the whole circle to any consonant part, it is also in consonance with the cut off part, just as by the previous proposition it was in consonance with the whole. If it is in the same proportion to it as the whole to some dissonant part, it will be in dissonance both with the cut off part and with the whole.

The first branch depends on Axiom V, as does one portion of the second branch also, that the remainder is in dissonance with the cut off part. However the proof that such a remainder is also in dissonance with the whole is as follows.

For it occupies, in the stated proportion, a position in the whole circle divided by an inconstructible figure. Hence although such a remainder is less than the whole circle, of which it is the remainder, yet it is greater than its semicircle, by the definition of a remainder. But if it is greater than its semicircle, then a quarter of its circle, that is half of the semicircle, is less than half of this remainder. Hence as the remainder is to a quarter of its circle, so will any circle divided by an inconstructible figure be to any part produced by its division. But such a whole circle is in dissonance with such a part of itself, by Axiom III. Therefore the remainder mentioned will also be dissonant with the quarter of its own circle, by Axiom V. Therefore it will also be dissonant with the whole of its own circle, by ***Proposition*** *VII.*

Corollary to these Propositions[52]

Therefore there are

Consonant Parts	Consonant Remainders	Dissonant Parts	Dissonant Remainders	In respect of the Whole
1	1	. . . . . .	. . . .	2
1	2	. . . . . .	. . . .	3
1	3	. . . . . .	. . . .	4
1 2	3 4	. . . . . .	. . . .	5
1	5	. . . . . .	. . . .	6
1 3	5	. . . . . .	7	8
1 3	. . . .	. . . . . .	7 9	10
1 5	. . . .	. . . . . .	7 11	12
1 3 5 . . .	. . . .	7	9 11 13 15 . . .	16
1 3	. . . .	7 9 . . .	11 13 17 19 . . .	20
1 5	. . . .	7 11 . . .	13 17 19 23 . . .	24

And so on.

[52] This table summarizes the conclusions of Kepler's axiomatic theory of the mathematical basis of the musical consonances. These may be expressed more succinctly in a simple formula; namely, the parts or remainders of a circle that are in concord with the whole are represented by *m*/*n* where *n* is the number of sides of a constructible polygon; *m* and *n* are co-prime and *m* is not the number of sides of an inconstructible polygon.

Chapter II.

On the Harmonic Division of the String.

So far we have described the origin of the harmonic proportions, that being two fold, one immediate from the constructible figures, the same being also congruent, the other through the mediation of double proportion, on which the identity of consonances depends. However since the harmonic proportions are infinite, being as far as our knowledge goes still rough, unpolished, unnoticed, and unnamed, and heaped together or rather scattered like some mass of rough stones or timber, the next thing is for us to proceed to polish them, to attach names to them, and finally to construct from them the splendid edifice of the harmonic system, or musical scale. Its construction is not arbitrary, as some may suppose, not a human invention which may also be changed, but entirely rational, and entirely natural, so much so that God Himself the Creator has given expression to it in adjusting the heavenly motions to each other. Now the harmonic proportions are fitted together with them into a single system by the harmonic divisions of the string. How many they are in number will be the subject for investigation in this chapter.

Definition

Extremes { ---- 3.2 Whole
Means 2.1 Greater part
— 1.1 Smaller part }

If the whole string is divided into parts such that they are individually in consonance both with each other and with the whole, we shall call the division harmonic. Now the middle term of this division, in musical (that is, consonant) proportions, is one of two equal parts, or if they are unequal, the greater of them: the outer terms of a consonant proportion are the other, or smaller part, and the whole string.

Let the geometer note the analogy of the divine proportion, that is the proportion of extreme and mean, in which the whole bears the same proportion to the greater part as the greater bears to the smaller.[53] *For what in this geometrical division is the same proportion, in our musical division is the same quality, which is called concord, consonance, congruence, or harmony. Beware however of assuming a consonance of the same kind, just as in the geometrical case the proportion is unique. The ancients did not mention this division in this sense, as they did not know the true cause of consonances; but we shall deal below with their division of the string.*

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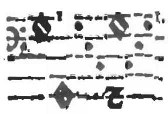
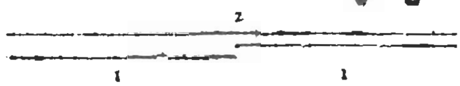
[53] This is the "golden section," which played an important role in Pythagorean music theory. See B.L. van der Waerden (1943), 163–199.

Proposition IX

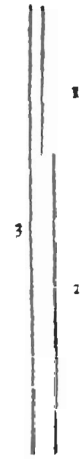
The division of a string into two equal parts is harmonic.

For because equal parts give out the same sound at any given tension, by Axiom II; and the whole is twice the individual parts; therefore it is in identical consonance with each one of them, by Proposition I. Therefore there are three consonances. Hence by definition the string is divided harmonically.





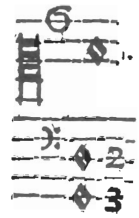
Notes to be expected in our usual music from what follows, for easier understanding.



Proposition X

The division of a string into two parts which are in double proportion is harmonic.[54]

For the parts in this proportion are in identical consonance, by Proposition I. And because the greater part is double the smaller, therefore the whole is three times the smaller. Therefore it is to the smaller as a circle is to the part cut off by a side of an equilateral triangle, which is consonant, by the final Corollary of the previous Chapter. Hence the whole is itself in consonance with the smaller part, by Axiom V. Therefore it is in consonance with the one which is double it, that is the remainder, by Proposition IV. Therefore three consonances are established by this division. Therefore the proposition follows.

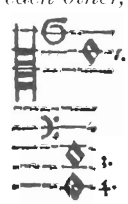


Proposition XI

The division of a string into two parts which are in triple proportion to each other is harmonic.

For because the parts 1 and 3 are to each other as a consonant part of a circle is to the whole, they themselves are also in consonance with each other, by Axiom V. And as 1 and 3 make 4, the part 1 will also be in consonance with the whole by Axiom I and by Proposition III.

Lastly because the remainder 3 is in consonance with the part 1, it will also be in consonance with four times the part, 4, that is with the whole string. Hence in this case also there are three consonances.



Proposition XII

The division of a string into two parts which are in quadruple proportion to each other is harmonic.

For because the parts are in quadruple proportion they are therefore in identical consonance with each other, by Proposition III; and because 1 and 4 make 5 therefore the part 1 is in consonance with the whole 5, by Axiom I and the Corollary mentioned. Hence the whole 5 is also in consonance with 4, the quadruple of the part 1, by Proposition IV. Therefore three consonances occur. Therefore, and so on.

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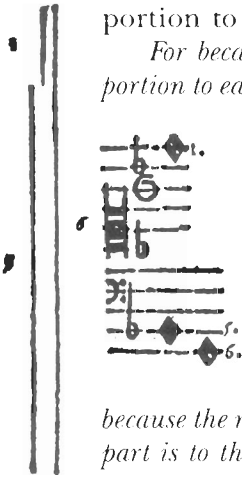
[54] The symbol 𝄡 denotes the C (soprano) clef, in which the bottom line is middle C.

Proposition XIII

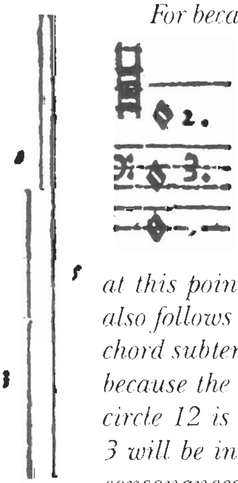
The division of a string into two parts which are in quintuple proportion to each other is harmonic.



For because the part is 1, the remainder 5, they are therefore in the proportion to each other in which the whole circle is to a consonant part, by Axiom I and the Corollary mentioned. Hence they are also themselves in consonance with each other, by Axiom V; and because the part 1 together with the remainder 5 makes up the whole, 6, therefore (by Axiom I and its Corollary) the part 1 is in consonance with the whole 6. And because the remainder 5 is to the quarter of the whole circle 6 (that is to say, to $1\frac{1}{2}$ in this division) as the whole circle 10 is to its part 3, which is consonant by the Corollary, hence the remainder 5 will also be in consonance with the whole 6, by Proposition VII. Or, which comes to the same thing, because the remainder 5 is to twice the whole circle 6, that is 12, as a consonant part is to the whole, by the Corollary, hence this remainder, 5, will also be in consonance with 12, twice the whole, by Axiom V. Therefore it will also be in consonance with the simple circle, that is to say the whole circle 6 itself, by Proposition IV. Thus three consonances occur. Therefore, and so on.

Proposition XIV

The division of a string into two parts, in sesquialterate proportion to each other, is harmonic.

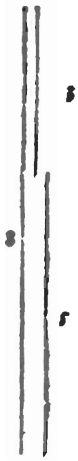


For because the part 2 makes the sesquialterate proportion with its remainder, 3, therefore the part is to the remainder as a consonant remainder 2 is to its circle, 3, by the Corollary. Hence this part 2 will also be in consonance with its remainder 3, by Axiom V; and because the part 2 together with its remainder 3 makes a whole 5, but a part 1 and its remainder 4 are in consonance with their whole 5 by the Corollary; therefore the whole 5 will also be in consonance with 2, which is twice its consonant part 1, which is our part at this point, or with 2 as half its remainder, 4, by Proposition IV. The same also follows directly from the axiomatic first part of Proposition V: because the chord subtended by two fifths is constructible, hence it is also consonant. Lastly because the remainder 3 of a part 2 is to a quarter of the whole, 5, as a whole circle 12 is to its consonant part 5, by the Corollary, therefore our remainder 3 will be in consonance with the whole 5 by Proposition VII. Therefore three consonances exist. Therefore . . .

Proposition XV

The division of a string into two parts in the proportion of one and two thirds to one, or 5 to 3, is harmonic.

For because the proportion of the part 3 to the remainder 5 is the same as that of any remainder 3, which is consonant by the Corollary, to the whole, *5, hence by Axiom V our part 3 will also be in consonance with our remainder, 5. And because the part 3 together with the remainder 5 makes a whole of 8, hence by the Corollary the part 3 will be in consonance with the whole, 8. Lastly because the remainder, 5, is to 4, the half of the whole, 8, as the whole circle, 5, is to a remainder, 4, which is consonant; or to the fourth part, 2, of the whole 8 as the whole circle 5 is to its part, 2, which is consonant by the Corollary, therefore our remainder will also be in consonance with its whole, 8, by Proposition VII. Therefore in this case also three consonances occur. Therefore . . .*[55]



Proposition XVI

If a string is divided into two expressible parts, and between them and the whole, that is between the three terms, there is one dissonance, there must also be another dissonance between them.

For the cause of the dissonance will be that either the whole or the part has from that division a number of portions which belongs to an inconstructible figure. But such a number is allied by consonance neither with any greater number, which belongs to a constructible figure, nor to any smaller than itself, by Axiom III and V and Proposition V and VII. Therefore the term which is made up of such a number of portions is in dissonance with the two remaining terms in that division; and thus there are two dissonances at the same time.

To this proposition the following proposition in geometry is similar, that if a straight line is divided into expressible parts, and one of them is incommensurable with a third (not with the whole made up of both of them as in this case) the other must also be incommensurable with the same third part.

Or, if a straight line is divided into parts which are incommensurable with each other, each will be incommensurable with the whole.

Proposition XVII

If a string is divided into two parts which are expressible in length, and there are two consonances between them and the whole, that is, between the three terms, there must also be a third consonance.

For if there are two consonances, since there are not more than three proportions, therefore there cannot be two dissonances. If there are not two dissonances, therefore there is not one either, by the converse of XVI. Therefore all three proportions will be consonances.

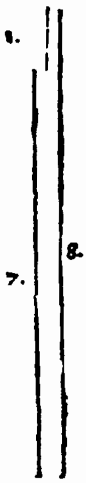
[55] The foregoing propositions established divisions of the string which produce all the consonances; that is, unison (1:1), octave (1:2), fourth (3:4), fifth (2:3), major third (4:5), minor third (5:6), major sixth (3:5), and minor sixth (5:8). The following propositions will show that there are no further harmonic divisions of the string. This is essential to Kepler's theory, for such further divisions would introduce dissonances.

In the same way in geometry, if a straight line is divided into parts which are commensurable with each other, the whole will be commensurable with both the parts.

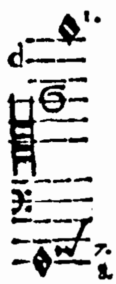
Proposition XVIII

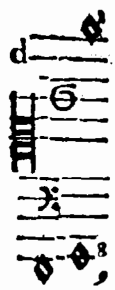
The division of a string into two parts which are expressible in length, in which either the whole or one of the parts acquires the number of portions which belongs to an inconstructible figure (where in fact the numbers both of the whole and of the parts have no common factors), it is not harmonic.[56]

It is proved like XVI. For at least two dissonances occur between the three proportions of the three terms, which is contrary to the foregoing Definition.

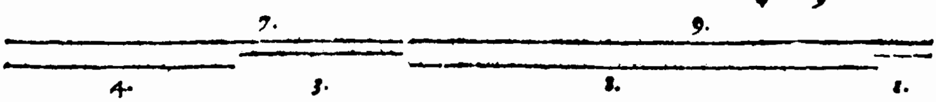


In this case there are three examples. In the first the greater part is seven eighths; in the last the smaller is one ninth; in the middle one, the whole contains seven parts. All are dissonant.





Places marked with a [note] cannot be expressed in the notes of the usual music.



Proposition XIX

After the octagonal, no harmonic division of a string is produced.

For the subsequent divisions either occur through inconstructible figures and their stars, and then although the parts may be consonant with each other, yet they are dissonant with the whole, by Axiom III; or through figures which are constructible by an inappropriate construction, such as the pentekaedecagon[57] *[fifteen-sided figure], and parts which belong to this division are in dissonance with the whole, by the Corollary to Axiom III; or through figures which are constructible by an appropriate construction, which after the Pentagon are*

[56] According to the definition at the beginning of the chapter, just one discord would be sufficient to conclude that the division is not harmonic. However, by proposition 16, there are two discords, and this, for example, enables Kepler to exclude 7:8 from the list of harmonies. For 1:7 is a discord because of the inconstructible heptagon and of the three ratios 1:7, 7:8 and 1:8, two are discords by proposition 16, but since 1:8 is a concord, it follows that 7:8 must be a discord. The other cases treated by Kepler enable him to exclude 3:7, 4:7, and 8:9 from the list of concords.

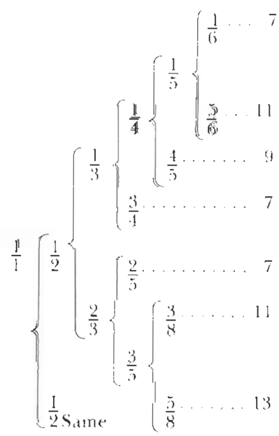
[57] It was also necessary for Kepler to exclude the 15-sided polygon, although this is in fact constructible, as this would have given rise to unwanted ratios.

all figures of an even number of sides: see Book I. Then the parts which belong to such divisions must consist of an uneven number of portions by the division; for if they were represented by an even number, the part would belong not to this division, but to a previous one. Thus if a string is divided into 10 and you take 4 or 6 portions, it is just as if you were to divide the string into 5 and take 2 or 3 portions. Then since the part is of an uneven number, the whole is of an even number, and the part can indeed be in consonance with the whole if it is not greater than fivefold (by Proposition V.) But one consonance is not sufficient for a harmonic division, as is evident from the definition. In that case then the remainder will be in dissonance; for the whole is assumed to have more than 8 portions and the definition of the remainder is that it is greater than half, that is to say greater than 4. Then the smallest remainder in an eightfold division is 5; in those of greater number it is greater than 5. Then in all the divisions of the string subsequent to the eightfold, the remainders are of uneven number, greater than 5. But uneven numbers greater than 5 belong to inconstructible figures, by XLV and XLVII of the first Book. Then by Proposition XVIII of this Book, these remainders bring about divisions which are not harmonic.

Corollaries

I. The harmonic divisions of a single string are seven in number, not more.

II. The expansion of the numbers which are characteristic of divisions occurs in the following manner. To begin with, the whole is expressed in the form of a fraction, that is to say with unity above as numerator, and unity below for denominator. Then each number separately is put as a numerator, and the sum of the two is put as denominator in each case. Hence from any given fraction two branches arise, until from the sum occurs the number which indicates an unconstructible figure. I I found these seven divisions of the string first with hearing as guide, in other words the same number as there are harmonies not greater than a single diapason. After that I dug out the causes both of the individual divisions and of the number of the total, not without toil, from the deepest fountains of geometry. Let the diligent reader read what I wrote about these divisions 22 years ago in *The Secret of the Universe*, Chapter XII,[58]



Causes of this sevenfold number found successively.

[58] The seven harmonic divisions were illustrated in exactly the same way as here in the *Mysterium cosmographicum*, Chapter 12. At that time, as he relates, Kepler attempted to derive the harmonies from the regular solids but later found the causes of the harmonies in the constructible polygons. Evidently he had expected to find inspiration in Ptolemy's *Harmonica* but when he was eventually able to read the work, he found that his own theory of the causes of the harmonies was wholly original.

and ponder how in that passage I was under a delusion about the causes of the divisions and the harmonies, mistakenly striving to deduce their number and the reasons from the number of the five regular solid bodies; whereas the truth is rather that both the five solid figures and the musical harmonies and divisions of the string have a common origin in the regular plane figures. Also by the generosity of Johannes Georg Heerward, Chancellor of Bavaria, I have obtained the *Harmony* of Ptolemy, together with the commentary of Porphyry, which I referred to in the passage mentioned; and from the third book of it I have transferred the more important part to the Appendix to Books IV and V of this work. Yet I did not find the true causes of the harmonies in them, and consequently no mention occurs even of these divisions and of their sevenfold number.

Unknown to Ptolemy and Porphyry.

Although I remarked at a fairly early stage that the causes must be sought in the plane figures, and you see the seeds[59] of the matter already scattered in the Chapter referred to, XII, of *The Secret*, yet they racked me exceedingly for a long time, before all my mind's doubts were satisfied. For first the constructible figures had to be separated from the inconstructible. Next I had to find the reason why although these divisions came from the figures, the divisions were restricted to seven but the figures extended to infinity. Thirdly, I had to establish the difference between the pentekaedecagon and the other constructible figures, because I saw that that figure was excluded from the begetting of harmonies, on the evidence of hearing.[60] Also the individual chapters had their own more limited hazards, each one of which kept me occupied for a long time. Take for example Proposition V, which I saw had to be added last of all, when I was already writing out the work, which I had not realized until then. For if that were not among the basic assumptions, and if, for instance, seven twentieths had on that account been suitable for setting up a harmony, because they are constructible through three twentieths (in combination with which they make a semicircle); and in that case both seven tenths and five sevenths, and so both two sevenths and one seventh, would be adjudged to make harmonies, which is in all respects rejected both by the ears and by our Axioms.

The sevenfold is established purely on the evidence of the hearing.

Therefore even by reference to the sole evidence of my book *The Secret of the Universe* the hearing is sufficiently fortified against the detraction of the sophists, and those who dare to disparage the trustworthiness of the ears on very minute divisions, and their very subtle discrimination of consonance—especially since the reader sees that

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[59] For example, he had stated that the perfect consonances (fourths, fifths, and octaves) come from the square and triangle of the cube, tetrahedron, and octahedron, but the imperfect ones (thirds and sixths) from the decagon of the other two solids. Since then, he had realized that the polygons were the causes of consonance in themselves and not by virtue of being surfaces of solid figures.

[60] See *Harmonice mundi*, Book I, Chapter 44.

I followed the evidence of my ears at a time when, in establishing the number of the divisions, I was still struggling over their causes, and did not do the same as the ancients did. They advanced to a certain point by the judgement of their ears, but soon abandoning their leadership completed the rest of the journey by following erroneous Reason, so to speak dragging their ears astray by force, and ordering them outright to turn deaf. Indeed I have taken extra pains below in Chapter VIII of this Book to ensure that anybody may have a ready opportunity to consult his hearing under his own colors on these and other divisions of the string, and of weighing up their evidence, so that he can be sure that we are struggling over the causes of what rests on the dependable test of the senses, and are not improvised fictions of my own (a charge of which the Pythagoreans stand accused) and intruded in the place of truth.[61]

[61] Once again Kepler emphasizes the empirical foundation of his scientific methodology, according to which hypotheses or causal explanations must be based upon observations.

Chapter III.

On the Harmonic Means, and the Trinity of Consonant Sounds.

It is superfluous to define harmonic proportion as that in which, three numbers being placed in their natural order, the amounts by which one of a pair of neighbors exceeds the other are in the same proportion as the outer numbers. Thus in the numbers 3, 4, and 6 the greatest, 6, is twice the smallest, 3; and similarly the difference, 2, between the two greater neighbors, 4, 6, is twice the difference, 1, between the two smaller neighbors, 4, 3.

Method of establishing any mean as musical in the opinion of the ancients.

However I shall include a method of finding numbers[62] which contain such a proportion, which is called musical by the authorities, because it is frequently transferred from the theory of harmony to ethics and politics. The method is as follows. *Given two numbers having no common factors, which contain the proportion both of the outer numbers (of three which are to be musically combined according to the scheme of the ancients) and of the differences of each from the mean. Multiply each by itself and both by each other. Of the three results add together the two smaller for the smallest of the numbers which are to be found; add together the two greater to find the greatest; and double the mean to find the musical mean of the ancients. For instance, let there be three numbers to be found in such musical proportion of the ancients that the outer ones are in the proportion of 3 to 5. Three times 3 is nine. Three times 5 is 15. Five times 5 is 25. Therefore the results are 9, 15, and 25. Add 9 and 15: the result is 24. Add 15 and 25: the result is 40. Twice 15 makes 30. Therefore the three required numbers are 24, 30, and 40. Their differences (of the outer numbers from the mean) are 6 and 10. Now as 3 is to 5, so 24 is to 40, and so also is 6 to 10. In the lowest terms which have no common factors, 12, 15, 20.*

Refutation of this method.

This indeed is truly a harmonic proportion according to me also, because not only is the proposed proportion between 3 and 5 harmonic, by the Corollary of Proposition **VIII**, but also the mean number found, 15, makes consonant proportions with the outer numbers 12 and 20 by the same Corollary. But this does not always occur.[63] For

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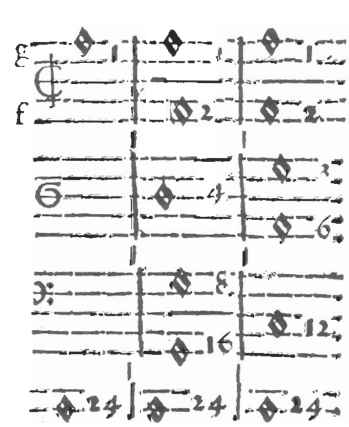
[62] Let the two numbers be a and b. Then Kepler calculates the harmonic mean of $a^2 + ab$ and $ab + b^2$ to be $2ab$.

[63] Although the harmonic mean, as defined by both the ancient and modern mathematicians, does produce some harmonic divisions, it also gives rise to divisions which are not harmonic. For this reason, Kepler interprets the term "harmonic mean" in a different sense, to denote the middle term in any unequal harmonic division of a string. Of the harmonic divisions described in Chapter 2, all but one are unequal, so that the number of means arising is one less than the number of divisions.

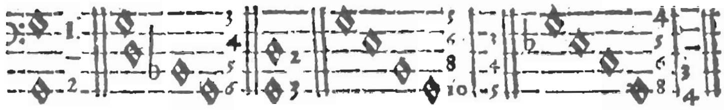
every time that the arithmetic mean, between two numbers proposed on this basis, marks out proportions with the outer numbers which are dissonant, there also emerge from this operation three numbers in a proportion which is in truth not harmonic, though the two originally proposed taken on their own form a proportion which is harmonic. That occurs in the case of 1 and 6, of 1 and 8, of 3 and 4, of 4 and 5, of 5 and 6, of 2 and 5, of 3 and 8, and of 5 and 8. For instance, between 2 and 5, that is 4 and 10, the arithmetic mean is 7, which is not harmonic, because 7 is not consonant either with 4 or with 10, by Proposition V. Then operate according to the rule. The resulting numbers will be 14, 20, and 35, with the excesses 6 and 15. Thus 20 ought according to the ancients to be declared the harmonic mean, because as 14 is to 25 (that is 2 to 5), so 6 is to 15. But the ears completely repudiate 20:35 (in other words 4:7) and 14:20 (in other words, 7:10).

Therefore in the harmonic divisions of Chapter II the number of means emerging is the same as the number of divisions, minus one. Also "mean" in those sections is in fact taken in its stricter sense, that in a string harmonically divided into unequal parts, it is the greater part, or the number expressing it. Thus 2 is the harmonic mean between 1 and 3; 3 is that between 1 and 4 and 2 and 5; 4 between 1 and 5; 5 between 1 and 6 and 3 and 8.

Apart from these there are also some other means, which are not subject to this law of the division of the whole string into two parts, but are included in our general definition, and divide not a single string, as in the previous Chapter, but the proportion of strings, into lesser consonant proportions.[61]

First, all proportions greater than double are resolved into their components, by the extraction of the double proportion. Thus 1:24 is made up of four doublings (that is, from multiplying by sixteen) and multiplying by one and a half. Hence as harmonic means under this heading 2, 4, 8, and 16 are interposed in this way between 1 and 24, taking the multiplication by 16 first, or 12, 6, 3, and 2 in this way, taking one doubling first and three later; for it can be done in various ways.

Secondly, a double proportion is resolved into the following consonances: 3:4 and 2:3, or 3:4 and 4:5 and 5:6, or 4:5 and 5:8, or 5:6 and 3:5. Lastly the sesquialterate proportion, 2:3, is resolved into 4:5 and 5:6. Similarly 5:8 is resolved into 5:6 and 3:4, and 3:5 into 3:4 and 4:5.

[61] Here Kepler applies his definition of harmonic mean to the further harmonic subdivision of a string.

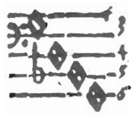
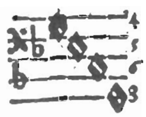
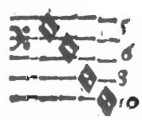
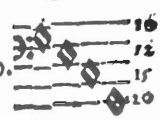
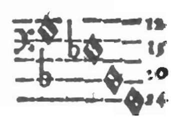
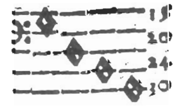


Therefore the three proportions 3:4 and 4:5 and 5:6 are the smallest of the consonances, that is, they are immediate, or without the harmonic mean, that is to say, they are consonant elements of the other proportions.

Harmonic mean twofold.

Now from that it follows that in one double proportion there can be two means, which are also consonant with each other, and in six ways. For because the double proportion has three smallest consonant elements, their order can be varied in six ways. For 3:4 is either in the first position on the smaller string, or in the middle place, or in the last; and in any given case, of the remaining elements either the greater with respect to the smaller string is 4:5, or the smaller is 5:6.

The individual cases have to be expressed in individual sets of four numbers, as shown in the following table.[65]

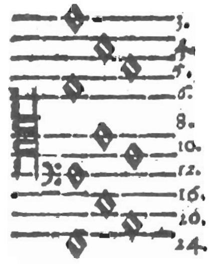
Order of the smallest concordant proportions in one double proportion. (Position of smaller or sharpest term.)		(Position of greater or deepest term.)	Group of pairs of harmonic means between strings in double proportion.	
3/4	4/5	5/6	3. 4. 5. 6.	
4/5	5/6	3/4	4. 5. 6. 8.	
5/6	3/4	4/5	5. 6. 8. 10.	
5/6	4/5	3/4	. . . . 10. 12. 15. 20.	
4/5	3/4	5/6	. . . . . . 12, 15, 20. 24.	
3/4	5/6	4/5	. . . . 15. 20. 24. 30.	

However among these six pairs of harmonic means, one alone, in the numbers 10, 12, 15, 20, comes within the definition of the ancients. For 12 is the musical mean (in their sense) between 10 and 15. Similarly 15 is the musical mean between 12 and 20. For the excesses are 2, 3, and 5. Now the outer numbers of one team of three, 10 and 15, are as 2 to 3; and the outer numbers of the other team of three, 12 and 20, are as 3 to 5.

[65] The development of the modern idea of musical chords proceeded in two stages. The first involved the recognition, above all by Zarlino, that three simultaneously sounding notes derived by a mathematical division of the fifth formed a unity. The second stage began with Johann Lippius, who introduced the idea of chord in-

On the Trinity of consonant sounds.

Then since strings in double proportion are in identical consonance, it is impossible for there to be between them in any one case more than two means, which are consonant both with each other and with their doubles. Hence arose that celebrated observation of the musicians, who wonder that all harmonies can be accomplished by three notes. For however many notes are assembled together additionally, each of them comes to the same as one of the three by the identical consonance of double proportion. For although one consonance emerges from strings of all these sizes—3, 4, 5, 6, 8, 10, 12, 16, 20, and 24—yet everything after the strings of length 3, 4, and 5 comes to the same as one of the following by identity of consonance: as 6 comes to 3 and 8 to 4 and 10 to 5; similarly 12 comes to 6 and 3; 16 to 8 and 4; 20 to 10 and 5; 24 to 12, 6 and 3.



The cause of this fact different people seek vainly in different ways: some in the threefold dimensionality of the perfect quantity, or body, as it appears in length, breadth and depth; some in the perfection of the threefold number; others in the revered Trinity itself of the Divinity.[66]

All, I say, vainly. For neither does three-dimensional quantity enter into this affair, since we have learnt that the origin of the harmonic proportions is in plane figures. Also three-dimensional quantity is greatly different, as far as knowledge of it is concerned, from two-dimensional, inasmuch as the former employs two mean proportionals, and in knowing them it is impossible for there to be any confusion; nor can there be any power in a number, insofar as it is considered as a counting number; nor, furthermore, is the origin of this trinity immediately from the Divine Being, causing it by imitation, as it has been made clear above that the cause of this matter is in the basic principles which were expounded, which in no way imply any particular number of notes on their own, but by fitting together individual notes to individual notes harmonically, and thus while doing something else accidentally produce something similar to the Divinities on account of the number's being the same. The same thing also happens in very many other matters.

In short, this threefold number is not the efficient cause of the harmonies, but an effect of the cause, or a concomitant of the har-

version. Kepler contributed nothing to this second stage, for he always regarded the base note as the reference note, a fact which to some extent restricted his concept of major and minor tonality. See Dickreiter (1973), 154. In modern terminology, the six chords described by Kepler consist of the major and minor chords in root position and in first and second inversion, but he regarded all of them as independent.

[66] Kepler rejected the idea of a theological significance in the unity of musical triads in terms of the Trinity, held by Cyriacus Schneegass and Johann Lippius among others, and insisted that the unity was only a mental concept based on the understanding. See Dickreiter (1973), 154.

mony which is effected. It does not give form to the harmonies, but is a splendor of their form. It is not the matter of the harmonic notes, but is an offspring begotten by material necessity. It is not the end "for the sake of which," but it is an eventual product of the work. Lastly, nothing results from harmony itself, but it is a secondary entity of the reason, and a concept of the mind, by second intention. For it is no more important to ask why only three notes are harmonically consonant, and a fourth and all others come back somehow or other to the same thing by the consonance of double proportion, than why there are only six pairs in any given octave, six forms of triple consonances. For as this sixfold does not come from the six days of Creation, so neither does that threefold depend on the Trinity of persons in the Deity. But since the threefold is common to divine and worldly things, whenever it occurs the human mind intervenes and knowing nothing of the causes marvels at this coincidence.

Chapter IV.

On the Origin of the Melodic Intervals which are Smaller than Consonances.

What is low?
What is high?
What is deep?
What is raised?

As sensation bears witness that of strings which are under equal tension the longer ones give lower sounds, and the shorter ones give higher sounds, hence these linked words, *high* and *low*, are the appropriate differences in harmony. For individually indeed they belong separately to other individual branches of knowledge, in which they are joined with other opposites: acute [the same word in Latin as high] with obtuse, in geometry; heavy [the same word in Latin as low] with light in physics. And in other contexts, sharp [the same word in Latin as high] expresses the meaning thin and penetrating; heavy [the same word in Latin as low] in matters of sensation is adopted for smells which, like heavy weights, on account of their magnitude are less bearable. But *high* and *low* linked together, and opposed to each other, are used only about musical notes. However, they keep some of their original meaning. For as in geometry acute is less than obtuse, so also in harmony a high note expresses the meaning small, and so penetrating and raised, in the Teutonic idiom, and fluttering as if raised aloft, on account of a certain lightness. And as in physics heavy things have a great weight, and light things a small, so also in this case a low note expresses the meaning "large"; and as things which are heavy in weight seek the depth and what is low, while light things soar up to the heights, so also in harmony, a low note on account of its magnitude is considered weighty, and so deeper or profound (*bass*), and a high note, as stated, is considered raised. For the fact that on the lute the first string, that is the highest, gave out a low sound was due only to its position on the instrument, as still today, and not on account of any similarity of the note to things which are light and fluttering aloft. However its position on the instrument has a mechanical reason, from the fact that the end string, that is the last and lowest, had to be struck most frequently because it gives the highest sound, as swift motion is convenient for small things, and indeed we strike downwards more readily than upwards on account of the shape of the thumb. A further reason is experience of the human throat. For not only in general are men dominant over women, and grown men over boys, and also give out a lower note, as if drawn from a greater depth, but also individual men, as we learn from the sense of touch, extract a lower note from a greater depth, and a higher from above; and those who sing lowest extend the body so that the note may emerge as deeply as possible. Indeed those who sing high also stretch their necks, yet not to make their necks long, but to tighten more effectively the upper bands of their throats.

Therefore it was for these reasons that there was born in harmony the concept of raised and deep, for which we frequently use high and low. Therefore since high and deep are at other times words referring to place, the habit of speech, following these its basic principles, also adapts to notes what properly belongs to places, that is to say intervals, in Greek διαστήματα, "separations." For it is places which are said to be at a distance, διΐστάναι, "be separated." Lastly, the discipline of harmony has also transferred this word to its pictorial representations or staves (which will be treated below), as they consist of high and low lines. On that basis the geometrical sense has been restored to the word.

What a diastema or interval is.

Therefore what were hitherto called the proportions of the strings will in future generally be called the intervals of notes which strings of unequal length give out. For notes of the same sound, corresponding with strings of equal length and equal tensions, do not make an interval, since they are of equal pitch.

Nevertheless in Book V this sense of the word "interval" will have to be avoided, because there the use of the word in its astronomical sense will have to be frequently repeated, referring to the straight line between the body of a planet and the Sun, and also to the space between different spheres.

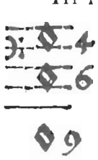
What a superior or an inferior interval is.

Furthermore in the preceding Chapter the proportions were considered under two headings, that is to say either individually or in their own right, or in relation to each other with respect to their order, which extended from the smaller term, or string, of some compound proportion, to the greater or longer, and the other way round. Now the intervals also are considered either individually or in their own right, or in relation to each other, with respect to their harmonic position. Thus in the continuous ordering of a number of intervals (as when every pair of adjacent intervals always has the same term in common, which is the greater term of one, and the lesser of the other which is in the direction of the lower notes) the interval which is between the lower notes is always called the inferior one, and the one which is between the higher notes the superior.

What equal intervals are.

And in geometry indeed proportions are recognized as equal even if the terms of one are not equal to the terms of the other, and the difference between the terms of one is not equal to the difference between the terms of the other. Thus if there are three strings in the proportions of the numbers 4:6:9, the proportion 4:6 is considered the same as 6:9 notwithstanding that both the actual terms, and also the differences 2 and 3 are unequal.

In harmony similarly all intervals between notes coming from the strings which are in the same proportion are both considered equal and are also written with the same numerical mark. Furthermore they are depicted on the stave with equal intervals of lines, so that we completely forget the inequality which there is between the differences of the various strings.



It therefore follows that we call the intervals which have a smaller proportion, smaller [in Latin "minor"] and those which have a greater proportion greater [in Latin "major"], without regard to the greatness or smallness of the corresponding terms in either case.

Therefore with these preliminary statements serving as definitions, we must now proceed to examine the differences between the intervals. So far indeed all the proportions which we have shown to be consonant, with the sole exception of equality, must be taken to represent the same number of intervals which are in like manner consonant; whereas the proportions which we have said to be dissonant represent a like number of dissonant intervals. However, there is a great difference between the dissonant intervals, so that not only are the consonant intervals taught to us by Nature and approved by hearing at her prompting, but other smaller intervals are also established by the same sense which although they are dissonant are yet suitable for conveying melody. Harmony, following Nature, attaches to them the name of melodic, and distinguishes them from the unmelodic, which have no place in the flow of any ordered melody. In Greek they are called ἐμμελῆ, "in melody" and ἐκμελῆ, "outside melody."

What are consonant, what are dissonant intervals?

What are melodic and what are unmelodic intervals, with reference to the name?

When the ancients saw this ingenuity of Nature in distinguishing between the melodic and unmelodic, they therefore thought they should try to find what was the smallest element common to the melodic and the consonant, by taking some number of which any consonance or melodic interval could be made up. For it seemed necessary that some such smallest interval should exist, as simple, and prior in origin to the consonances themselves, which seemed to be made up of such a smallest element, inasmuch as some intervals were larger than others.[67]

Yet the reality is far different, as can be learnt from many examples. For if in all species the individuals, which differ in size, are made up of one common smallest element, therefore there will be some single smallest quantity of the human species; and from some definite number of striplings of that kind, as if from elements, any man you like may be composed, a lofty one from many, a dwarf from few. For in harmony the quality known as consonance shapes the proportion of the strings, or the interval of the notes, just as much as the shape of a man shapes that mass of matter which is surrounded by a man's skin. And why did they forget geometry, in which there are a great many examples of every kind of incommensurable quantities, which are defined as sharing no common measure whatever, which belongs to quantities of the same kind, as a definite quantity of some element of their composition.

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[67] It was probably more natural for the Greeks, whose recognition of consonances was rather limited, to regard the melodic intervals as the fundamental elements out of which the consonances could be built. On the grounds that consonances are in general incommensurable, Kepler rejects this idea, claiming instead that the consonances are the fundamental intervals from which the melodic intervals are derived.

Most musical intervals incommensurable.

Therefore we must accept that consonant intervals (except in cases where one is a multiple of the other) are, like the actual proportions, incommensurable, in such a way indeed that although their differences may be expressed in numbers, which in simple numbers is a sign of commensurability, yet these differences, not of course being simple numbers but fractions, are not an aliquot part or aliquot parts of the differing terms, in relation to any number. For instance the two proportions 1:2 and 1:4 are to each other as the number 1 to the number 2. They are, then, commensurable, for 1:4 is twice 1:2. There is room for this in the series of continuous doubling alone.[68] For in the series of triples, and the other multiples, two consonant proportions do not occur. Thus 1:9 is indeed triple 1:3; but only 1:3 is among the consonances, and 1:9 is among the dissonances, by Axiom III. We can see the same thing in non-multiples. Thus in the case of the sesquialterate proportion, 2:3, a consonance, its multiple certainly occurs and is thus commensurable; for 4:9 is to 2:3 as the number 2 is to 1, but 4:9 is not among the consonances. On the contrary two others are consonant, as in the series of continuous doubles, like 1:4 and 2:3. These two proportions are not commensurable with each other, that is, they are not as number to number; for the excess of 1:4 over 2:3, 3:8, is not measurable by either 1:4 or 2:3.

Therefore the consonant intervals are by nature prior to the smaller intervals which we name melodic; and they are not composed of melodic intervals as if of elements, or of some smaller quantity, but on the contrary the melodic intervals arise from the consonances, as if from causes.

At this point we must take note that the word "composition" is ambiguous. Sometimes it denotes the natural origin of a thing, sometimes however the quantitative division of a thing, which is not an origin, but rather a destruction, as when we say that a circle is made up of three thirds, first mentally dividing the circle into three, or when we say that the human body is composed of members, not because the members existed before the body, and the body was assembled and constructed from them, as a house is from stones and wood, but because the body in virtue of its bulk is divisible into these members, which separately and independently are no longer a functional body.

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[68] Kepler uses the word proportion with two meanings. Here it is used in the sense that 1:4 is double the proportion 1:2 and the proportion 1:9 is triple the proportion 1:3. But he also uses the term in the modern sense, in which, for example, the double or triple proportion of another proportion is its square or cube, while the half or third part is its square root or cube root. Thus, for example, the half of the double proportion is equal to $\sqrt{1:2}$, where the term is used in both senses. When Kepler speaks of the sum of two proportions he means the product and when he speaks of the difference he means the quotient. Also Kepler regards the smaller of two proportions as that which is closer to 1, so that, for example, the proportion 1:3 (or $\frac{1}{3}$) is greater than the proportion 2:3 (or $\frac{2}{3}$)

In the former sense we must say that consonant intervals are composed neither of other consonances nor of melodic intervals. In the latter sense the consonant intervals, which are larger, certainly do consist, and are thus in a sense composed (as we ourselves have assumed previously) of the smaller consonances, and the smallest consonances of melodic intervals, and so on, because they are analyzed into these elements, so to speak; but the various intervals among themselves do not consist of some larger number of intervals of a single very small kind, common to each other, and cannot be analyzed into any such.

However, the consonant intervals also have related causes, yet they do not all have the same cause, but each its own special cause, distinct from the others' causes, as has been explained previously. For consonance is a property of the actual intervals, not according to their quantity directly, nor directly according to their relationships, but according to their relationships qualitatively (that is, in a sense, as figured). Thus to seek to establish a smallest interval which is common to them is inappropriate, since smallest and greatest are observed not in qualities but in bare quantities and in their proportions, whereas to divide consonances, as consonances, is to destroy a kind of consonance, and in its place to establish either other kinds of consonance, or dissonant melodic intervals, or even downright unmelodic intervals. An interval therefore does not take the causes or elements of its consonance from parts as if they were basic principles, in the same way as commensurable quantities are built up by the multiplication of a common measure, and along with that measure belong to one and the same type. On the contrary, what the ancients took to be the basic principles of consonances (tones, I mean, and semitones and dieses[69]) originate from the consonances as their true basic principles.

The type of a consonance as such does not arise from the number of melodic intervals as parts.

For although the consonances do consist of these melodic intervals which are not consonant (if not from a single one in common, at least from several combined with each other in various ways), yet that must not be attributed to the actual consonance of the interval. For if the melodic intervals imparted to a larger interval, which was composed of them, its own consonance, that would always occur in any multiple of melodic intervals, and the more melodic intervals there were in it, the better would be the consonance. However that is false, for as we shall hear below, two tones combined make a consonance, three combined make up a dissonant interval.

Nevertheless the fact that a consonance can be analyzed into dissonant melodic intervals, as will follow, is clearly accidental to that consonance considered on its own, and occurs only insofar as several

[69] Thus all the tetrachords in the diatonic scale were composed of semitone, tone, tone, those of the chromatic scale of semitone, semitone, three-semitones, and those of the enharmonic scale of diesis (quartertone), diesis, ditone (two tones).

consonances are compared with each other, each originating from its own basic principles.

Definition of a melodic interval.

Therefore melodic intervals are defined as being all the differences between consonances which are smaller than the double interval;[70] and the natural faculty of hearing does not admit any other intervals as melodic but those which arise from this subtraction. Thus the consonant intervals take their origin from geometry and the constructible figures, but the melodic intervals from the actual consonances; and the melodic intervals stand in relation to the consonances just as in geometry the Apotomae (inexpressible lines) stand to those expressible in square, for the former are also defined by subtraction of an expressible line from an expressible line.*

The origin of melodic intervals in consonances.

Furthermore there is one method of comparison or abstraction which is general or arithmetic, and another which is special, and proper to harmony. But it is by arithmetical means that consonances which are less than double are selected, so that one of them is not a part of another, in the sense of being indicated by some harmonic mean, as in the previous chapter.

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[70] That is, the octave.

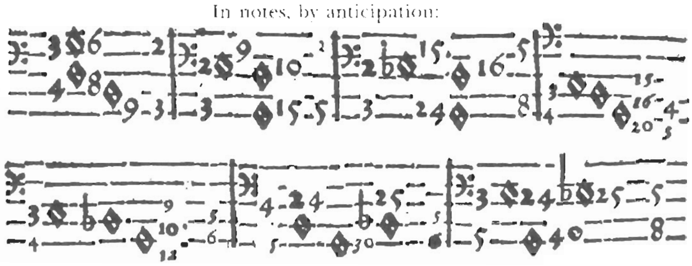
* *The text has been assembled from sheets written at various times, but not well enough fitted together; and although everything in it is true, it has nevertheless produced obscurity by confusing what ought to be distinguished and by repeating propositions. The distinct questions are as follows.* I. Do the consonances have parts, which are themselves also consonances, or at least melodic intervals? *This is the answer:* In consonances there is a distinction between, first, the proportion, which is something geometrical, and, secondly, the quality of the proportion, the consonance itself. Therefore insofar as they are proportions, one is exceeded by the other, the smaller by the greater, and thus one can be a part of the other; but insofar as any of them has received the quality of consonance from its own constructible figure, they do not have the property of being compound. The arguments are as follows. 1. A kind as a kind is one and indivisible. 2. A kind is established by its cause; but the causes of individual consonances are distinct from each other. Therefore the consonances themselves are distinct from each other in kind, and one, as the greater, cannot be analyzed into others of its own kind, as parts, but can into smaller ones of another kind. 3. If, as a part when multiplied increases the quantity, in the same way a consonant or melodic part of a consonance increased the consonance or melodicity of the whole, dissonances and unmelodic intervals would not eventually be produced by accumulation.

Question II. Do the consonances have one common measure, I mean a tone, diesis, comma, and so on. *The answer is negative:* for it is so neither inasmuch as they are proportions, because in that respect they are incommensurable, that is have no commensurable measure, nor inasmuch as they are kinds, because by the causes from which they arose and by which they are defined, they are also mutually distinguished from each other. For they have in a sense the nature of figures, in view of the fact that a triangle and pentagon in the same circle have sides of incommensurable lengths. Lastly, a measure is after all prior to what it measures; whereas the melodic intervals, such as a tone, diesis, and so on, are posterior to the consonances. And note that parts such as a major or minor tone, semitone, and so on, are common to consonances, but not all to all consonances, and there is none of them which is a complete measure of consonances on its own.

There are between

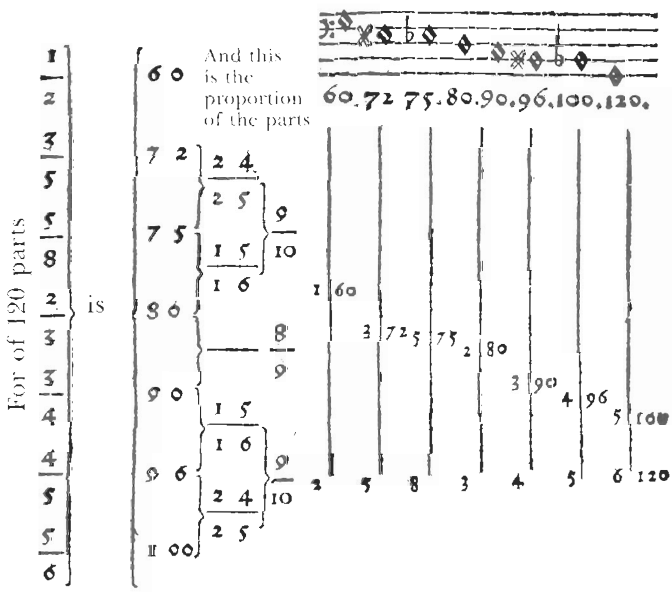
These concords	Melodic intervals
$\frac{2}{3}$ & $\frac{3}{4}$	$\frac{8}{9}$
$\frac{2}{3}$ & $\frac{3}{5}$	$\frac{9}{10}$
$\frac{2}{3}$ & $\frac{5}{8}$	$\frac{15}{16}$
$\frac{3}{4}$ & $\frac{4}{5}$	$\frac{15}{16}$
$\frac{3}{4}$ & $\frac{5}{6}$	$\frac{9}{10}$
$\frac{4}{5}$ & $\frac{5}{6}$	$\frac{24}{25}$
$\frac{3}{5}$ & $\frac{5}{8}$	$\frac{24}{25}$

In notes, by anticipation:



The harmonic comparison of consonant intervals refers to their origin, and to the degree of height which is assigned to each of them on account of its origin. For every greater term in the comparison of proportions is represented by one and the same whole circle, and the complete string which is analogous to it, as common to all harmonic divisions. Therefore we have to find for all the numbers representing the greater terms in the seven harmonic divisions, that is to say 2, 3, 4, 5, 6, 5, 8, a lowest common multiple, 120; and the whole string must be divided into the same number of equal parts, so that the sound of the whole string is established as the common greater term of all the consonances made by the divisions, and the smaller terms must be fitted in such a way that when set along-side each other they set up the melodic intervals, which are investigated in this Chapter. However the results are the same as before, arithmetically.

Shown here in notes, by anticipation.[71]



[71] Kepler uses the sign ♯ both to indicate a sharp note and also to indicate a natural note after the cancellation of a flat. In this table, the sign in both cases is equivalent in modern notation to ♮, indicating the natural note.

This, therefore, is the origin of the dissonant melodic intervals, to which we shall give their names a little later on.[72]

On the generation of third intervals.

The next thing is for us to speak also of the origin of third intervals, which although they are not exactly melodic, yet are useful in melody, and serve the purpose of melodic intervals.[73] Now they arise from the subtraction or comparison of melodic intervals (in the same way as melodic intervals from those of consonances). For between melodic intervals or seconds are the following third intervals:[74]

$\frac{8}{9}$ and $\frac{9}{10}$ . . . . $\frac{80}{81}$

$\frac{8}{9}$ and $\frac{15}{16}$ . . . . $\frac{128}{135}$ Which is composed of 24:25 and 80:81 and is a very little less than 15:16.

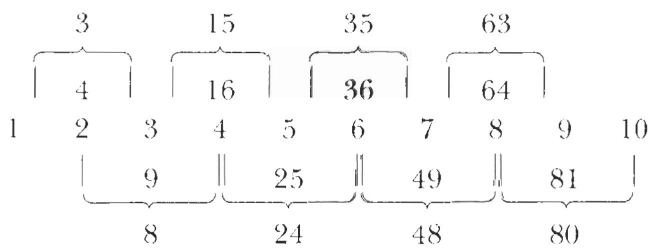
$\frac{9}{10}$ and $\frac{15}{16}$ . . . . $\frac{24}{25}$

To those can be added twice the interval 15:16, or in other words 225:226, which is greater than the interval 8:9 by a little less than 15:16 is greater than the interval 128:135. Now the first three indeed arise from the mutual division of various melodic intervals; but this last from the addition of two melodic intervals which are equal, but in a less common way.

Hence springs to notice a very splendid

Arithmetical corollary

in numbers and in the following diagram:



That is to say, the square of every number below ten, together with the rectangle having the two numbers which are its closest neighbors

[72] The reader may find it more convenient to have the names at this point. They are major tone (8:9), minor tone (9:10), semitone (15:16), and diesis (24:25).

[73] These third intervals, or differences of melodic intervals, will be found useful in modulation.

[74] The comma (80:81) is the difference between the major tone and the minor tone. The limma or major diesis (128:135) is the difference between the major tone and the semitone; it is slightly smaller and scarcely distinguishable from the semitone. The diesis (24:25) on the other hand is the difference between the minor tone and the semitone, and in Kepler's view, is of such imperfection (being the progeny of two imperfect intervals), that it almost ceases to be worthy of the name melodic. Singers usually overshoot it, except in modulation.

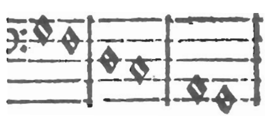
as sides, makes up either a consonance or a melodic interval or a third interval, with the exception of 49, the square of the sevenfold, and its two rectangles 35 and 63. But here the melodic interval 9:10 is banished, and most of the consonances, except 3:4. It is therefore fortuitous, depending on the order of the numbers and the structure of this diagram.

In vain will the arithmetician seek causes from this direction, in vain will the Pythagorean be obsessed by his fascination with the sevenfold, as a counting number: the matter must be pursued more deeply in geometry, and in the counted and figured numbers, that is to say in the inconstructible figures themselves, of which the first is the heptagon. For what prevents the possibility of the diagram's being continued beyond ten, with the nature of the melodic interval following, is now no longer the sevenfold, but the other numbers belonging to inconstructible figures, 9 and 11, which make the rectangle 99, which with 100, the square of the tenfold, makes up an interval which is totally abhorrent to the nature of melody. So great a difference is there between axioms of conjecture and axioms of knowledge.

The order of the melodic intervals in perfection, and their names.

We have spoken so far of the origin and order of the intervals which are less than consonances. Now we must also speak of the differences between them, and of their names, which we could not keep absolutely the same as the ancients, since we shall have to differ from them on these matters and on their causes.

Therefore it is in agreement with what has been said above, especially Axiom II, that of those intervals which belong to the nature of the melodic, each one retains the nature of the consonances from which it is established. Therefore, since of the consonances which are smaller than the double interval, the most perfect are 2:3 and 3:4, on account of the nobility of the figures from which they take their origin, their joint progeny among the melodic intervals also, that is to say 8:9, must be elevated above the others. We shall therefore give this interval, in common with the ancients, the name WHOLE, and on account of this preeminence, we shall call it a perfect tone, but on account of the size of the proportion, a major tone.

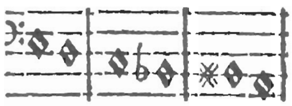


What a major tone is.

On the other hand if you compare the greater perfect interval 2:3 with the greater imperfect interval 3:5 on the higher side, or the smaller of the perfect intervals 3:4 with the smaller of the imperfect intervals 5:6 on the lower side, there will be born from this marriage the rather imperfect melodic interval 9:10. It is smaller than 8:9; and since that interval in ancient music before Ptolemy[75] was generally not men-

[75] Ptolemy introduced the minor tone especially into his scale called "diatonon syntonon" (tense diatonic), where each tetrachord is composed of a semitone, major

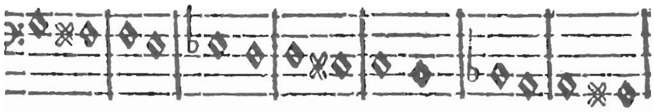
A minor tone.



tioned, inasmuch as the theoreticians demonstrated all intervals through the full tones previously defined, we shall give it the name of minor or little tone, so as to mark its imperfection. Here let the reader take note, in a word, that some have given that name to another interval, so that if he happens by chance to read them he may not be caught off his guard and confused.

Yet if you link the greater perfect interval 2:3 with the smaller imperfect interval 5:8 on the higher side, or the smaller perfect interval 3:4 with the larger imperfect interval 4:5 on the lower side, the melodic interval arising from the comparison, that is to say 15:16, again brings in

A semitone.



an element of imperfection from the fact that this is its origin, and will be called a semitone,[76] the same term as is used for this interval in normal present day music, because it is a little larger than half a major tone. Some have maintained that it is a minor tone; but the reader should be wary of them, so as not to be confused.

These three, therefore, arising from perfect intervals by comparison of them either with each other or with imperfect intervals, have acquired the property of being melodic in their own right and always.

On the other hand if you compare with each other the imperfect intervals arising from the pentagon or decagon, either on the higher side 3:5 with 5:8, or on the lower side 4:5 with 5:6, the interval arising from it, that is to say 24:25, is of such imperfection that it almost ceases

A diesis.

to be counted as melodic. This interval we shall call a diesis, using the ancient word, as if to say, a slackening of the string. Nor am I at pains to propose by this term the same size of interval as the ancients: and again let it suffice to give notice of that.[77] There are three causes of its imperfection: origin, small size (since it does not equal a third part of a perfect tone), and because it is also listed among the third intervals above, that is to say among those which are serviceable for making tuneful kinds of melody. For it also arises from the comparison of the minor tone and the semitone. Now this interval is not melodic in its own right nor always; for the human voice does not usually pass over this interval in one and the same ἀγωγῇ, "approach," as it does the other intervals, but it leaves it out and overshoots it, with the sole exception of a modulation in the melody, to

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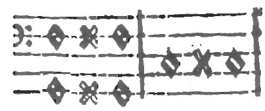
tone and minor tone. *Harmonica*, Book II, Chapter 1. This was the scale that Zarlino supposed singers of his day actually used.

[76] Here the sign ※ indicates a sharp note.

[77] Here the sign ※ indicates the natural note. The Greeks took the diesis or minor semitone (also the Platonic limma) to be 243:256. See Boethius, *De institutione musica*, Book II, Chapter 28.

add flavor. Then it becomes extraordinarily melodic, but in such a way that it begins, so to speak, a new kind of melody; and it requires art and no little toil to achieve that with the human voice without an instrument. Thus this interval only marks the difference between kinds of melodic intervals, and is serviceable for them on that basis.

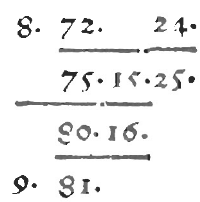
We have begun to speak of third intervals; for the same interval which is the first of them, 24:25 or a diesis, was also the last of the melodic intervals. There now follow the designations of the remaining ones. For 128:135 which arises from 15:16 and 8:9 can be designated a major and irregular diesis. As stated above it is a very little (that is to say by the amount of 2025:2048) smaller than the melodic interval of a semitone and is scarcely distinguished from it. And under this title it is among the melodic intervals, because it plays the part of a legitimate diesis, particularly at a modulation in the melody. For its genesis is both natural and necessary in practice, so that semitones and dieses are available in all directions, for the sake of various flavors of melody. For that reason, when a legitimate semitone is split off from a major tone, and this interval remains, we can designate it too by a Greek name, limma or remainder.



A major diesis or limma.

Finally we can call the difference, 80:81, intervening between 8:9 and 9:10, a comma, in Latin a segment or cut. For the ancients cut their diesis into four parts, and hence called them commas, believing that this was the common element of all consonances. Now this interval is a little larger than a fourth part of our diesis, and smaller than a third. For 24:25 is 72:75 or 96:100. Therefore a third part would be 74:75, and a fourth 96:97, about; and 80:81 is between the two. We could define a comma by a closer number as an eighth part of a major tone, that is to say 8:9. That is also clear as follows. The major tone 8:9 is divided into a diesis, 72:75, a semitone, 75:80, and a comma, 80:81. Now a comma was just found to be about a third of a diesis. Therefore about four commas are equal to half a tone, and eight to a whole tone, approximately indeed, not absolutely. Therefore this interval is plainly not among the melodic ones which are sung in succession, because their small size is scarcely perceptible by the hearing, still less expressible in human melody on their own independently, by two notes in succession. But it does not cease to be melodic, like 11:12 and similar intervals, because we are also comparing things which are separated in space and time. However a double semitone must be established because in the division of tones which succeed each other in order, two semitones are sometimes placed in succession; and people occasionally use them combined together as a tone when they are aiming at variety and novelty, to express grave disturbances of the mind.

A comma.



A double semitone.

Note that between the semitone 15:16 and the diesis 24:25 there is 125:128, roughly 42:43 or a double comma. If you add to that a comma,

A double comma.

A triple. Narrow diesis. Platonic limma. Wide semitone. Plato's apotome.

80:81, the result is 625:648, roughly 27:28 or a triple comma. However the same 80:81 subtracted from a diesis, 24:25, leaves 243:250, which is as nearly as possible 35:36. The same comma subtracted from a semitone 15:16 leaves a Platonic limma, 243:256, which is roughly 19:20; but added to 15:16 it makes 25:27 which is between 12:13 and 13:14. Thus two major tones, 8:9 combined together make 64:81; and it was by dividing that interval into 3:4 that Plato established his limma. However on subtracting 243:256 as a limma from a major tone Plato had as a remainder 2048:2187, which he named an apotome;[78] and it is larger by one comma, 80:81, than our limma, 128:135, and exceeds a semitone, 15:16 very little.

Although these are abnormal intervals, nevertheless mention will be made of some of them below in Book V.

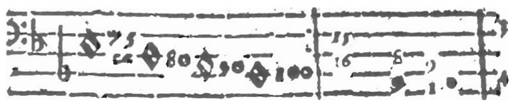
[78] The Platonic limma or minor semitone together with the apotome make up a major tone. Boethius, *De institutione musica*, Book II, Chapter 30. The relationship of the small intervals to the whole tone is the principal subject of Boethius' Book III.

Chapter V.

On the Natural Division of Consonant Intervals into Melodic Intervals, and their Designations which Arise from That.

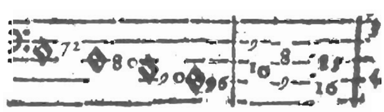
What the melodic intervals were, that is as observable by human ears in the course of a melody, and imitable by the voice of a singer, was stated in the preceding Chapter. Now we must examine with particular care into what melodic intervals each of the consonances may be analyzed, with Nature as guide.

Again, therefore, taking the same numbers by which all the natural harmonic divisions of a single string had been brought together under a single point of view, we shall clearly see that the interval[79] between the numbers 75 and 100, with the



interpolation of 80 and 90, has resulted in the following three melodic intervals: the semitone 75:80 or 15:16, the major tone 80:90, that is 8:9, and the minor tone 90:100, that is 9:10.

The same has also happened with the interval between the numbers 72 and 96, with the intervention of the same numbers, 80 and 90: for 72:80 is 9:10, a minor tone; and 80:90, as above, is a major tone; and lastly 90:96



is 15:16, a semitone. Now on both sides between the outer terms, both 75:100 and 72:96, the interval 3:4 is detected. Hence as Nature has taught us to fit these numbers together, through the division of the circle by constructible figures, therefore Nature has split these two sesquitertiate intervals, in a definite location between two terms in double proportion, by the very fact of the divisions, into three perfect melodic intervals, a major tone, a minor tone, and a semitone. But for three neighboring intervals there must be four positions or notes or strings. Hence, therefore, the sesquitertiate interval has come to be called a fourth, with the implication that it is the fourth note from the first, whether above or below. For the same reason the Greeks call this interval Διά τεσσάρων ("over four"), which we also express in Latin letters, writing in the normal way diatessaron.

A fourth or diatessaron.

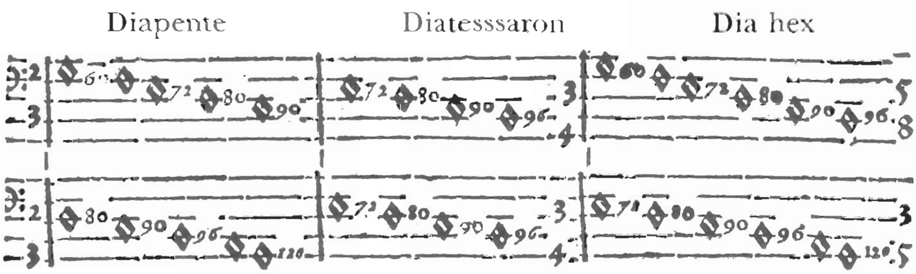
[79] This is the division of the tetrachord (fourth) adopted by Ptolemy for the scale that he calls "diatonon syntonon." *Harmonica*, Book I, Chapter 15.

A fifth or diapente.

It follows therefore that because the sesquialterate interval adds one extra perfect tone (inasmuch as the difference between 2:3 and 3:4 is 8:9), from that fact it is called a fifth, or, in imitation of the Greek expression, a diapente ("over five"), notwithstanding that sesquialterate intervals are not split by the very fact of our harmonic divisions into that number of melodic intervals. For that reason we are still short of one number which is necessary for this complete split; and God the Creator Himself has also expressed this defect in the planetary motions, as we shall hear in the fifth book.

Sixths, major and minor.

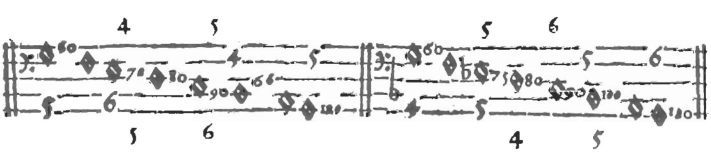
Further because in the same way both 5:8 and 3:5 also add as extra to the sesquialterate interval, 2:3, one of the elements already mentioned, the former in fact a semitone, 15:16, but the latter a minor tone, 9:10, from this fact they are designated sixths, the former in fact minor, and the latter major.



On the other hand because both 4:5 and 5:6 take away from the sesquitertiate interval one of the elements already mentioned, the former in fact a semitone 15:16, but the latter a minor tone, 9:10, as was revealed in the preceding Chapter, therefore there remain to each of them no more than a pair of the melodic elements, to the former in fact tones, major and minor, to the latter a major tone and a semitone. From this fact some designate these intervals by the Greek word ditones,[80] major and minor or semiditone; and since the two intervals require three terms or notes, they are therefore called thirds, major and minor. And of these intervals established by the natural divisions of the string the remainder are actually split up by these means in the following way, but the highest and lowest have not yet been split up by these means.

What a major and a minor third is.

Galilei uses the name of ditone to mean something else, from very ancient music, which differs from the major, consonant third. See Chapter XII.



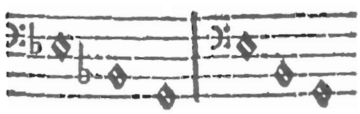
And since the harmonic ratios 3:5 and 4:5 come from the penta-

[80] The Greek ditone, consisting of two major tones, exceeds Kepler's ditone by a comma. The Greek ditone, to which Vincenzo Galilei gave the same name, was an element of the enharmonic scale. See Boethius, *De institutione musica*, Book I, Chapter 21.

gon, of which the side is inexpressible, and both 5:8 and 5:6 have some admixture of the pentagon, hence it comes about that both pairs are of rather imperfect consonance.[81] The less it is, the softer and smoother it sounds to the ears. Now it is less in the case of 5:6 and 5:8, because they divide the whole circle either by a more perfect figure (that is to say the hexagon, the side of which is expressible in length) or into parts in the proportion of continuous doubling (which is identical), that is to say into 6 and 8. Therefore 5:8 and 5:6 are taken to be the soft sixth and third; but 3:5 and 4:5 the hard or harsh sixth and third; and they are indeed designated in that way.

Origin of the names hard and soft third and sixth.

Soft Hard



Lastly, because the interval of double proportion, as has been stated above, consists of a sesquialterate proportion and a sesquitertiate proportion, of which the former is called a fifth, the latter a fourth, but both of which have in between them one common term, which is the last of one and the first of the other, in the same direction in the following way:

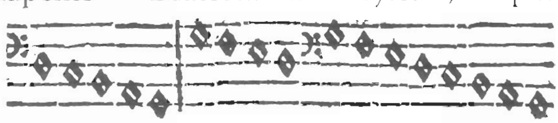
An octave or diapason.

First.	Second.	Third.	Fourth.	Fifth	6.	7.	8.
				First.	Second.	Third	Fourth.

It therefore follows that the last of the one which is later in the order of counting is the eighth in number. Hence this interval has acquired the name of octave. The Greeks having regard to identical consonance designate it Δια πασων ("over all") which is also written in Latin letters diapason, as if having completed all the notes which sound different, the voice of the singer returns to itself at the eighth note. Hence a new start, a new series of melodic intervals arises, similar in all respects to the previous one. See Proposition I.

What an octave or diapason is.

Diapente Diatessaron System, Diapason



Do people vainly philosophize at this point about the number, that is to say about why the eighth note completes them all, and returns to the same? For in truth the answer must be through a circular argument, because, that is to say, it comes about naturally that the interval

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[81] It follows that, on theoretical grounds, the minor intervals are less imperfect than the major ones and according to Kepler, this is in accordance with the evidence of the sense of hearing.

of double proportion, which by Chapter I is identical in sound, is divided into seven melodic intervals, which are bounded by eight notes, as has been proved in this Chapter. They think that comes about because the number 8 is the first cubic number and the first cubic shape. But what has the division of a string to do with solids? And why does not the twenty-seventh note also (that being the second cube) return to the same point?

What is a system?

Hence the name of system definitely belongs properly and primarily to the interval of the double proportion, split up into its seven melodic intervals, and set out in its eight notes or strings, and expressed on instruments. That will be dealt with below, in Chapter IX.[82]

Disdiapason.

In the cases of those intervals which exceed an octave or diapason, in fact multiples of an octave, the double octave is usually called in Greek Δὶς διὰ πασῶν (dis dia pason, "twice over all"), the triple Trisdiapason (thrice over all), and so on. In other cases however the diapason is expressed, or the multiple of it, and the excess is added, on the following basis: fifth above the octave, or octave above the fifth, diapasonepidiapente, or Διὰ πέντε ἐπὶ διὰ πασῶν [Dia pente epi dia pason –"over five in addition to over all"]. Sometimes we also proceed by a number, with the designations ninth, tenth, twelfth, and so on.

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[82] The principal Greek system extended over two octaves. In the basic element of Greek music, the tetrachord (spanning a fourth), the end notes were fixed but the inner ones movable. There were three genera of scale, the diatonic, the chromatic, and the enharmonic. In the diatonic genus, the steps upward in each tetrachord were semitone, tone, tone. In the chromatic genus they were semitone, semitone, 3-semitones, and in the enharmonic genus they were diesis, diesis, and ditone. In the case of each genus, there were also several species, arising from variations in the definitions of tone, semitone, and diesis.

A scale greater than an octave was needed to set the octave in different keys or modes. The scale over two octaves may be illustrated as follows, where the fixed notes are shown in heavy type and the tetrachords are indicated by brackets.

A H c d **e** f g **a** **h** c′ d′ **e′** f′ g′ **a′**

Thus each octave consists of a whole tone followed by two conjoint tetrachords. This was known as the diezeugmenon (disconnected) system, because the octaves were separated by a tone. Ptolemy, *Harmonica*, Book II, Chapter 5. Ptolemy (ibid., Chapter 6) observes that the ancients also used another system in which the first whole tone was followed by three conjoint tetrachords, as shown below.

A H c d **e** f g **a** b c′ **d′**

This was known as the synemmenon (connected) system.

The practice of designating notes by letters of the alphabet was introduced by the Greeks. Pope Gregory formalized the system by using the first seven letters of the Roman alphabet. The first series, A, B, . . . G, are called graves, the second, a, b, . . . g, acutes and the third, a′, b′, . . . g′ or aa, bb, . . . gg, super-acutes. In agreement with Kepler, we follow the German convention of designating the notes making a minor and a major third respectively with g by the letters b and h.

In the eleventh century Guido d'Arezzo introduced a tone Γ below A in the Greek system (see note 98 for the reason). It became customary to take the note Γ (or G), indicated on the lowest line of the base stave as the fundamental note. Kepler follows this convention. The letters should not be taken to imply an absolute pitch.

Therefore this division of consonant intervals into melodic intervals is natural, and the size and number of the elements, which is not greater than threefold, does not depend only on the habituation of the ears; but the hearing takes that from natural instinct. Nor can either other intervals be taken, apart from those, or another number of intervals, as melodic intervals into which any of the consonances may be split up. For if you wanted for example to include the diesis, in the first place it is also the offspring of dissonant melodic intervals; and in the second place if you particularly wanted to include only it, because it is the offspring of consonances, yet you could not include it alone, for it would drag in with it the limma or minor semitone, which arises only from dissonances. But it is agreed that the hearing distinguishes the former from the offspring of consonances, and that if they are admitted among the singable, and among the second intervals, it repudiates the latter as spurious. Hence the diesis too will not be among the principal intervals of the octave.[83]

[83] This rules out the enharmonic genus of scale. The enharmonic scale, however, had been almost abandoned in the time of Aristoxenus. When people tried to sing it, he remarked, they got the intervals wrong and approximated to the form of the chromatic scale. See Barker (1984), pp. 183 and 244–246.

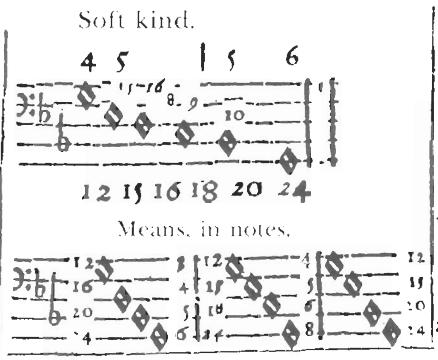
Chapter VI.

On the Kinds of Melody, Hard and Soft.

We have spoken of the kinds of figure in Book I, Proposition XLIX. Since they are imitated even by the actual divisions of the string, by Axiom II of this Book, it therefore follows that as the division in the proportion of continuous doubling, and the triangular division, and the continuous doubling of it, are among the first at least of the figures with expressible sides, the triangle and quadrilateral, whereas the pentagonal division is found to have its side inexpressible, therefore the former divisions by Axiom IV produce one kind of melody, the latter the other kind. With this will be included indeed bisection also, not on account of the tetragonal figure but only on account of the identical concord formed by bisection.

From this, therefore, are born two kinds of divisions, and one in fact has the following divisions:[81]

Divisions	Common denominator	Intervals
1/2	12	4/5
5/8	15	15/16
2/3	16	8/9
3/4	18	9/10
5/6	20	5/6
	24	



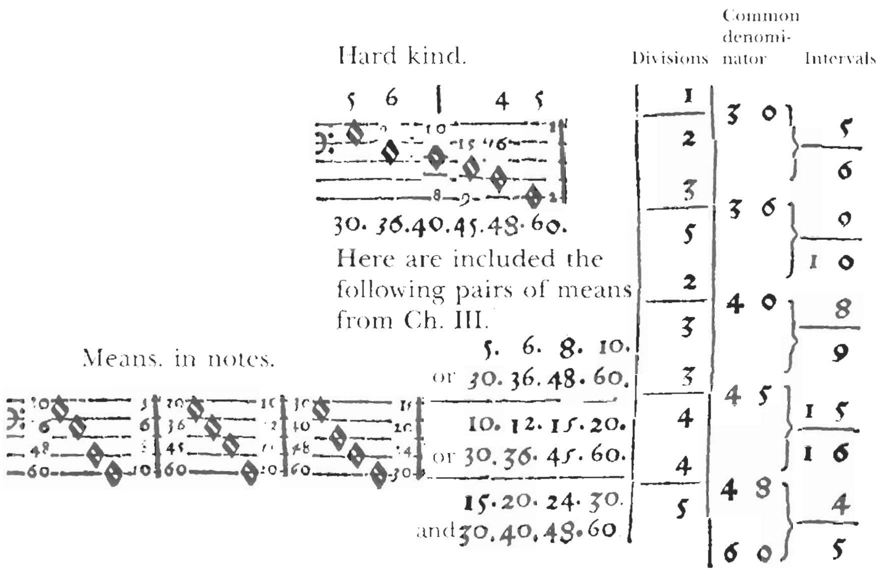
In this kind, of the six pairs of means in Chapter III the following are admitted.

3. 4. 5. 6.
or 12. 16. 20. 24.

4. 5. 6. 8.
or 12. 15. 18. 24.
and 12. 15. 20. 24.

The other kind embraces the following divisions:

[81] As the divisions 3:5 and 4:5 come from the pentagon, the hard kind, containing these divisions, is less perfect, according to Kepler's principles, than the soft kind.



These are the popularly celebrated two kinds of melody; and the former in fact is called soft melody, because the intervals found in it, in order from the lowest note, the third and sixth, are soft, but the latter hard melody, from the intervals of the same name which are in the same position in the system of the octave in order. The reason for their name has been stated in the preceding Chapter, V.

For just as in the former case 5:6 in the lowest position did not accept 3:5, so now in the latter case 4:5 does not accept 5:8, because the nature of tuneful melody requires that the third with the sixth makes a perfect fourth or diatessaron.

From this, therefore, the natural boundary of the two kinds is apparent. For since in soft melody 5:6 is in the lowest position, in the hard 4:5, and the difference between them, that is to say the diesis, 24:25, is not one of the ordinary melodic intervals, by Chapter IV, then in the same sequence of natural melody 4:5 and 5:6 cannot stand at the same time; but if 4:5, a major third, is inserted, in that case 5:6 must be expelled from the lowest place, or if the latter is accepted the former is ejected. Then 4:5 draws with it 3:5, and 5:6 draws with it 5:8, a minor sixth.

Now again the distinction between the two kinds of harmony has been expressed by God himself in the motions of the planets, as we shall hear in Book V.

Of the ancients' three kinds, of which the names are diatonic, chromatic, and enharmonic,[85] I here deliberately refrain from speaking, so as not to confuse the reader. However you may understand by diatonic, hard melody, by chromatic, soft melody; or by diatonic the separate kinds, but by chromatic a mixture of hard and soft. The enhar-

[85] See note 82.

monic however has nothing corresponding to it in natural melody; but in normal music there correspond to it up to a point the vibrato of the human voice, the tremolo of instruments, the throbbing in the strings of the pandora, and the like.

See our arguments on these kinds below in the Appendix to the text of Ptolemy.[86]

[86] As already remarked, Kepler did not in fact include his translation and commentary on Ptolemy's *Harmonica*, Book III in the published work. The arguments to which he here refers are those in his commentary on Ptolemy's *Harmonica*, Book III, Chapter 6 (**KOF**, vol. 5, pp. 355–357). In this chapter Ptolemy compares the individual genera of harmony with different kinds of virtues. Kepler disagrees with Ptolemy in principle, as he explains in his appendix to Book V of the *Harmonice mundi*. Whereas Ptolemy sees causal or natural relations between the harmonies and the virtues, Kepler claims that such symbolic associations are merely poetic and rhetorical. Concerning the Greek genera of harmonics themselves, Kepler does not believe that they had a natural origin. For example, the diatonic genus was at first influenced by the method of tuning instruments by successive subtraction of the major tone. See Aristoxenus, *Elementa harmonica*, 55. This made the scale hard and against nature. Singers, on the other hand, instinctively approximated to the natural system that Kepler describes in Chapter 7. For this reason, Kepler believes, the Greeks introduced the chromatic genus, which softened the hardness of the diatonic and was in some sense nearer the natural scale of the singers. But the chromatic fails in certain ways and it was for this reason, Kepler concludes, that the Greeks intentionally put together the enharmonic genus for softness.

Chapter VII.

On the Complete Division of One Octave in Each Kind of Melody, and on the Natural Order of all the Melodic Intervals.

Therefore Nature has so far shown that in soft melody the third melodic interval from the bottom is a minor tone, 9:10, the fourth a major tone, 8:9, and the fifth a semitone, 15:16, whereas in hard melody the third from the bottom is 15:16, the fourth 8:9, and the fifth 9:10.

There still remain in each kind of melody two thirds, respectively minor and major, which have not yet actually been split up by natural divisions of the string, into their smallest melodic elements.

Let us first see what melodic intervals they can be divided into, and then what order they should individually be placed in.

Now nature teaches[87] clearly enough that if we can split them into the same intervals of which we have hitherto on Nature's showing seen concords of diatessaron to consist, we should not use any others of which Nature does not show examples. Therefore arithmetic teaches that 4:5 consists of 8:9 and 9:10, and similarly 5:6 of 8:9 and 15:16. See, these are the same melodic intervals as hitherto.

Now unless you place in the lowest position of the lower third in each case a major tone, it will not be possible for a single string to play both kinds of melody. For if in soft melody you cannot place a major tone in the lowest position, therefore you must place there a semitone, 15:16, because these are the only two included in the undivided interval 5:6. In hard melody, on the other hand, a minor tone would have to be placed in the lowest position, because the former concord undivided, that is to say a major third, does not have in it a semitone, which previously had to be placed in the lowest position in soft melody. Thus there would be two strings, the longer of which would establish a semitone, with the longest string, for soft melody, and the other a minor tone for hard melody.

Secondly must be added the fact that it seems in agreement with Nature for the larger intervals, whenever we have a free choice of division, to tend towards the low notes, because the low notes themselves are also fuller than the high notes.

By the same primary reasoning we shall also carry the point that the upper third, hitherto undivided, must be divided in such a way that the major tone is in the highest position, to avoid making the seventh string twofold. For in soft melody 4:5 is above, in hard melody

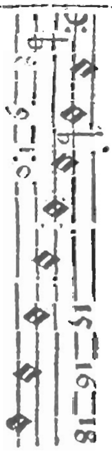
[87] Thus the major third is divided into a major tone and a minor tone, while the minor third is divided into a major tone and a semitone.

5:6, by the natural method imparted above. Therefore if we were to place a perfect tone (that is to say the other element of a third) in the lowest position in order within this upper third, then the same interval arising from notes of different pitch would also extend to notes of different pitch which it would make into two instead of one. These logical and plainly necessary arguments are sufficient against the authority of Ptolemy, Zarlino, and Galilei, who have a minor tone in the lowest position of the octave.[88]

Therefore the eight strings are exhibited in the following numbers, reduced to the same lowest common denominator.

The System of the Octave

In soft melody.

Sounds or positions	In notes	Length of strings		Upper	Fourths: Mean in the natural division		Fourths: Lowest
			Vel			Vel	
VIII.		72.	360.	24			
VII.		81.	405.	27			
VI.		90.	450.	30	30.	15.	
V.		96.	480.	32	32.	16.	
IV.		108.	540.		36.	18.	27.
III.		120.	600.		40.	20.	30.
II.		128.	640.				32.
I.		144.	720.				36.

[88] Kepler has misinterpreted Ptolemy, Zarlino, and Galilei. In Ptolemy's "diatonon syntonon" (tense diatonic), which Zarlino and Galilei in effect adopted (see note 31), the intervals were as follows.

A H C D E F G a

T s T t s T t

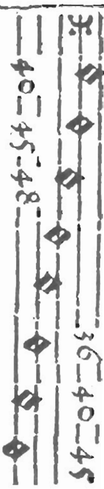
where T = major tone, t = minor tone, and s = semitone. If Ptolemy's scale had started on G, the first interval would have been a minor tone, as Kepler claims, but it started on A with a major tone. It should be noted that, in the Greek synemmenon system (as in the diezeugmenon system), the first interval is a major tone, so that the second note is H. Consequently, the semitone step from a to b can only occur in the higher octave. In Kepler's scale for hard melody, the intervals were as follows.

G A H C D E F g

T t s T t s T

The reason that Kepler gives for adopting this arrangement—namely to avoid introducing two extra strings—seems to be quite clear.

In hard melody.[89]

Sounds or positions	In notes	Length of strings	Fourths: Upper	Fourths: Mean in the natural division	Fourths: Lowest
VIII.		360.	120.		
VII.		405.	135.		
VI.		432.	144.	36.	
V.		480.	160.	40.	
IV.		540.		45.	135.
III.		576.		48.	144.
II.		640.			160.
I.		720.			180.

You see expressed in the smallest numbers not only the proportions of all the eight lengths, but also those of the four highest and the four lowest in smaller numbers. The most important, however, are those of the three highest and the three lowest. On the means of these and their numbers I have a controversy with the authorities which comes into this Chapter, for VII in both kinds of melody is 405, and II in each case 640.

Remember, however, that only the eight principal strings of each of the kinds are included in this Chapter. The subsidiary strings, one of which hard melody admits in place of the one before the highest, we shall examine in the following Chapter; for our aim here is to see how those next to the highest and next to the lowest have to be established so that they can be the same in both kinds.

[89] Kepler's scales in hard and soft melody (with the modification introduced in the next chapter for the hard kind) are equivalent to the Aeolian and Ionian modes introduced in 1547 by Henricus Glareanus, which have remained as the modern minor and major scales respectively. Kepler, like Zarlino, however, also classified the traditional modes as soft or hard, according to whether they contained minor or major thirds and sixths. Kepler's genera soft and hard relate more to the melodic concept of the traditional Church modes than to the harmony-oriented music of the seventeenth to the nineteenth century. Yet by emphasizing the fundamental distinction of two types of tonality, he anticipated the modern concept more strongly than Zarlino. See Zarlino, *Istitutione harmoniche*, pp. 182 and 210.

The traditional modes were obtained by taking different notes as the starting point of the central octave of the two-octave scale. See Ptolemy, *Harmonica*, Book II, Chapters 8–11. On Kepler's concept of tonality, see Dickreiter (1973), 160–170.

And because in the reconciliation of these eight principal strings of both melodies it comes about that two of them are double in musical instruments, therefore in the common system ten principal strings occur in one octave, but no more than eight POSITIONS are recognized (and are designated accordingly). See the origin of the term in Chapter IV.**

** See also the following diagram, which amounts to the same thing:

Positions in soft melody:	VIII	VII		VI	V	IV		III	II	I
Strings:	360	405	432	450	480	540	576	600	640	720
Positions in hard melody:	VIII	VII	VI		V	IV	III		II	I

Chapter VIII.

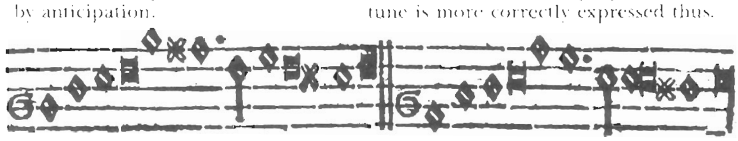
On the Number and Order of the Smallest Intervals in One Octave.

The lowest and highest intervals, and also the fourth, the latter by nature, the two former in imitation of Nature, have been made major tones; and from them, again in imitation of Nature, which divides the minor tone into a semitone and a diesis, are split off the same number of semitones, 15:16, for the sake of greater variety, especially in the twists and turns of melody, and that in the upper part of the interval. There remain therefore in the lower part limmas, or major dieses, 128:135, which exceeds the diesis 24:25 by one comma.

However, there is particular necessity for this splitting in the highest interval, which in the preceding Chapter we made a major tone, because hard melody uses a semitone in the highest interval or a major tone promiscuously and according to circumstances, but the former more often and almost regularly. Just as the ordinary seventh position makes with the fourth an epitriton or diatessaron 3:4 (that is to say 405 with the 540 of the previous Chapter), similarly this extraordinary number VII would love to make a perfect diapente or 2:3 with III, because III, that is to say 576, belongs to the hard kind, so that on this basis the melody becomes brighter.[90] And indeed $\frac{2}{3}$ of 576 is 384, which with VIII, that is to say with 360, makes 16:15.

Take this example in notes, by anticipation.

Though I would not dispute it if someone were to assert that the property of this tune is more correctly expressed thus.



But on this matter more below from what has been declared.

Thus there occur in one octave overall thirteen strings, in the following numbers or lowest terms. I have inserted among them all the smallest intervals according to the natural order in a full and perfect functional system.[91]

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[90] The modification of the hard scale which Kepler describes here was in fact the variant used almost universally in practice. For it was generally agreed that large and ascending intervals made melody brighter. Besides the reason that he gives for its use—namely that the tritone between notes III and VII (an interval difficult for singers and generally avoided) was thereby converted into the perfect consonance of a fifth—Kepler could have been attracted to this form of the hard scale because it agrees better with his theory of the harmony of the motions of the planets.

[91] Each major tone (8:9) is divided into a limma (128:135) and a semitone (15:16). Each minor tone (9:10) is divided into a semitone and a diesis (24:25).

	Lengths of strings	*Melodic or quasi-melodic intervals*
Above	1 0 8 0	
		Semitone
	1 1 5 2	
		Limma
	1 2 1 5	
		Semitone
	1 2 9 6	
		Diesis
	1 3 5 0	
		Semitone
	1 4 4 0	
		Semitone
	1 5 3 6	
		Limma
	1 6 2 0	
		Semitone
	1 7 2 8	
		Diesis
	1 8 0 0	
		Semitone
	1 9 2 0	
		Semitone
	2 0 4 8	
		Limma
Below	2 1 6 0	

In normal notes

Instrumentalists' deficient technique.

You see that in two places pairs of semitones occur together below a diesis, as we said above would happen. And on instruments the practitioners seize on that, and use two semitones instead of a tone, whenever they introduce some less common modulation. Yet from the practice of their craft they temper all intervals, in such a way that none is left true, but so that the intervals which ought to be perfect, by a minimum loss of perfection, mitigate and alleviate the imperfection of the rest. By this means it comes about that for the instrumentalists all the tones are equal, and the limma is equal to a semitone, and hence they arrange that two semitones make a tone, in a perfect bisection; and so as to achieve that more easily, not even the diesis retains its mathematical perfection at their hands.

Convenient division of the monochord for the lute and its rejection by the sense of hearing.

This point in the argument demands that I should fulfill what I promised above at the end of the second Chapter, namely that I should compare with my findings the division of the monochord for the lute proposed by Vincenzo Galilei, if I am not mistaken, from Aristoxenus;[92] and in addition that I should also interrogate the ears, yet in such a way that Reason should proclaim what it has agreed are the decisions.

For now the diapason consists of seven melodic intervals, from which it gets the name octave, as was stated in the previous Chapter, and between these intervals there are five tones, and two semitones, and any of the tones can be bisected into parts which are nearly semitones, so that twelve semitones are made, defined in this Chapter by the letters *G. Gϱ. A. b. ♮. c. cϱ. d. dϱ. e. f. fϱ. g.* Hence instrumental musicians have taken the opportunity to establish twelve intervals in the diapason themselves as well, and that indeed in the way which is least laborious but completely easy and convenient, as a mechanical method should generally be. It is as follows. They divide the whole length of the neck of the lute, in fact between the two housings or bridges, on which the string rests so that it can vibrate freely with the whole of the length between them, into 18 equal parts; and they place the first tie or fret from the nut so that there are one part above and 17 below. Then removing the points of division they divide these 17 remaining parts afresh into another 18, and with a second tie they divide off one part as before; and they repeat that twelve times. At the twelfth division they say that the part of the string left is of such a length that between it and the whole there is the consonance of diapason.[93] Then let the length of the string be 100000.[94] If 18 parts are 100000, what are 17? The result is 94444. Again if 18 parts make 94444, what are 17? The result is 89198. And if 18 make 89198, what are 17? The result is 84242, and so on. On this basis twelve numbers result for the same number of inter-

	Galile's argument	*True argument demonstrated up to this point*
G	100000	100000
Gϱ	94444	93750
A	89198	88889
b	84242	83333
♮	79562	80000
c	75242	75000
cϱ	70967	71111
d	67025	66667
dϱ	63301	62500
e	59785	60000
f	56463	56250
fϱ	53325	53333
g	50363	50000

Expressed in the same form of numbers.

[92] Aristoxenus (*Elementa harmonica*, 56–57) supposed the tetrachord to consist of two and a half whole tones, so that the octave was made up of 6 whole tones or 12 semitones. Vincenzo Galilei (*Dialogo*, pp. 49–53) adopted a system of equal temperament, in which the octave was supposed to be composed of 12 semitones with ratio 17:18. Although Galilei recognized the system to be an approximation, he defended it in his *Discorso intorno alle opera di Gioseffo Zarlino* (Venice, 1589), 109–118, for some Aristoxenian friends against the attacks of Zarlino, only to dismiss it in favor of the system of just intonation, which he supposed singers to use, with the promise of another work in which he would explain how the problem of instability was resolved in practice. See Walker (1978), 17.

[93] Since $(17/18)^{12}$ is not equal to $\frac{1}{2}$, the 12 semitones cannot make up an exact octave. Boethius (*De institutione musica*, Book III, Chapter 3) made a similar calculation to that of Kepler in order to refute the division of the octave into 6 whole tones proposed by Aristoxenus, showing that 6 whole tones (8:9) exceed the octave by a comma (524288:531441).

[94] In the column for Galilei, *c* should be 75142, *d* 67024, *e* 59784, *f* 53326, and *g* 50364. In the column for Kepler, *Gϱ* should be 94815.

vals; and alongside are set out the numbers for the lengths demanded by the true argument demonstrated up to this point, as parts of the whole string.

If this argument had been exact, the proportion repeated twelve times would be equal to the double proportion of 1:2, which good arithmeticians know to be false, since 1:2 and 17:18 are incommensurable. However this mechanical division of the string does somehow or other satisfy the hearing, on account of the fact that the individual numbers are close to the true values placed next to them, and because the strings of lutes can stretch, and somehow various notes are sharper to begin with, when the motion of the strings is still great, as they have just been released by the finger, and lower and more relaxed when the amplitude of the vibration is reduced to a narrow width as the string reverts to normal.[95] Furthermore there is a difference in a gentler and a stronger touch, and in a wider or thinner one, according to the discipline of music. Yet if you examine the judgement of the hearing with the skillful investigation of the reason, a disagreement will at once be evident, which I prove as follows. The ears certainly acknowledge harmony between 1000000 and 50363, as the mechanist asserts; and they acknowledge the same between 100000 and 50000, as I assert. The question is whether the ears notice no difference, and whether there is the same amount of latitude for the latter harmony. I reply, from my harmonic divisions of Chapter II, that they do not. For just as they detect no difference between the two consonances of strings of 100000:50000 and strings of 100000:50363, in the same way they will detect no difference between the two consonances of strings of 100000:50000 and strings of 100000:49637, because the difference between the two consonances is equal in each case.

But if each part is not struck along with the whole string, 100000, but the parts themselves, 50363 on one side and 49637 on the other, are struck along with each other, then it is absolutely necessary for the shorter part to emit a higher note, and the longer part a lower; and this difference between the notes is easily discerned by the ears, weighing parts in unison. It is therefore clear that the ears judge the consonance between 50363 and 100000 and between 49637 and 100000 to be the same for no other reason than insofar as 50363 and 49637 seem to be equal, that is, insofar as neither the one nor the other is sensibly different from 50000. Therefore if the ears reply according to the questioning of reason, they will say that the exact harmony is between 100000 and 50000, that is between 2 and 1. Hence it is false that the consonance between 100000 and 50363 is exact.

The verdict on the harmony between 100000 and 67025 is the same. For although at first the ears detect no difference from 66667, the cor-

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[95] Kepler is of course mistaken in supposing that the pitch changes with amplitude.

rect length of the truly consonant part, yet the remainder from the former, 32975, is not half of 67025. Therefore twice the former, 65950, will make a different consonance from 67025 since these strings compared with each other make a different note. Therefore the reason why in harmony the ears detect no difference between them is that they approve the mean number, if the sense is sharpened by reason, that is to say 66667, so that its complement is 33333, that is exactly half of it.

The sharpness of the sense of hearing having now been confirmed in discriminating the harmony of 2 to 3, and also of 1 to 3, the next thing is for us to make the same judgement over 2 and 3 with 5, and over 1 and 3 with 4. For if 59785 were in consonance with 100000, and so also its remainder 40215, certainly the latter is not two thirds of the former; yet the perfect consonance of the two thirds has already been vindicated, that is to say of 60000 with 40000. On this basis the hearing is carried across, if it is sharpened by reason, from one division to the other, starting from unison, which is the easiest to recognize and discriminate, right up to the point where it covers them all, approving the parts which my figures represent, and repudiating the mechanical figures in the first column.

See another very clever tempering of this sort by Vincenzo Galilei,[96] made not in ignorance of the mathematical size of the notes, but with a particular intention. And I indeed recognize its mechanical function, so that in instruments we can enjoy almost the same freedom of tuning as can the human voice. However for theorizing, and even more for investigating the nature of melody, I consider it ruinous; and the effect of it is that the instrument never truly attains the nobility of the human voice.

[96] While Zarlino (*Sopplimenti musicali*, pp. 140–149) favored just intonation without qualification, which he called "sintono naturale," Galilei (*Discorso*, 124), recognizing the problem of instability in this system, finally opted for what Zarlino called the "sintono artificiale" scale of Ptolemy, in which some intervals are flattened by a comma (80:81). Zarlino associated the natural scale with the voice and the artificial scale with instruments. Galilei denied the distinction between voices and instruments and also the idea that any scale was more natural than another. See Walker (1978), 19–23.

Chapter IX.

On the Stave, that is the Modern Representation of the Strings or Sounds, by Lines and Letters of the Alphabet and Notes; and on the System.

Lines.

The matter is not uncontroversial; but I am following necessary reasoning. The musicians of our age depict the strings in one way in staves for lutes, and in another for singing. For in the case of lutes the individual lines of the stave represent individual strings, each of which gives out various notes according to the various stops, if it is struck: in the diagrams for singing, not only do the individual lines signify individual notes or strings, but the actual intervals between two lines signify one intermediate string.

Strings.

On this basis four transverse lines with three intervals represent seven strings,[97] and five lines with four intervals, nine strings. Next, they express the notes in the melodic system either by special symbols, each in its own place on the stave, the symbols at the same time intimating a measure of time, or by the first seven letters of the alphabet, or, lastly, by the six syllables[98] *ut, re, mi, fa, sol, la,* or as the Belgians do today, by the following seven: *Bo, ce, di, ga, lo, ma, ni.*

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[97] The stave of 4 lines was used for plainsong.

[98] This is the nomenclature of the hexachordal system introduced in the eleventh century by Guido d'Arezzo. It enabled singers to keep their tonal bearings when sight reading. In this system, the vocal range was represented by a set of overlapping hexachords, each consisting of six diatonic notes with a single semitone between the middle pair. The names were taken from the opening syllables of the lines of the first verse of a Vesper hymn for the Feast of John the Baptist by Paulus Diaconus:

Ut queant laxis
resonare fibris
mira gestorum
famuli tuorum
solve polluti
labii reactum
Sancte Ioannes.

The alternative (Belgian) names were introduced for their sonority by Hubert Waelrant (1518–1595), who founded a school of music in Antwerp.

The addition of a tone below *A*, to which Guido gave the name Gamut (Γ) enabled him to convert the tetrachord hypaton into a hexachord in which the semitone occupied the central position. He added two tones to each of the other tetrachords, thus converting them into identical hexachords. The relationship between Guido's hexachords and the tetrachords of the Greek systems are shown in the tables below. It should be noted that, from Guido's point of view, the Greek synemmenon system was an

For because there are two kinds of melody, hard and soft, which start from one common lowest note, and the difference between them is that the third and sixth in hard melody are one diesis higher than in soft, thus the third and sixth strings cannot remain unaltered in both melodies, but must either be made double in instruments and organs, as they were in numbers at the end of Chapter VII, or varied by stopping as I believe was once done on the lute and on the pipes. The same must be said of the stave and of the human voice. For the octaves of both kinds, rising from the same point of origin, there is only one stave, and in it only eight positions, indicating eight notes. However, from the two kinds of melody there are ten principal strings squeezed together into one octave, by Chapter VII, but thirteen of all kinds, by Chapter VIII. The result is therefore that two notes, separated by a diesis or limma, are represented in the same position on the stave, whether it is a line or an interval between lines, because

Notes.

The necessity for letters.

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auxiliary system suitable only for progression through the note b. Guido classified the hexachords as hard, soft, or natural, according to whether they progressed through ♮, *b* or avoided these notes. For example, the hexachords beginning on Γ or *g* are hard, that beginning on *f* is soft and that beginning on *c* is natural.

DIEZEUGMENON SYSTEM

Tetrachord		*Greek Name*	*Hexachords*			
	a'	Nete hyperbolaeon				la
Hyperbolaeon	*g'*	Paranete hyperbolaeon				sol
	f'	Trite hyperbolaeon				fa
	e'	Nete diezeugmenon			la	mi
Diezeugmenon	*d'*	Paranete diezeugmenon			sol	re
	c'	Trite diezeugmenon			fa	ut
	h	Paramese			mi	
	a	Mese		la	re	
Meson	*g*	Lichanos meson		sol	ut	
	f	Parhypate meson		fa		
	e	Hypate meson	la	mi		
Hypaton	*d*	Lichanos hypaton	sol	re		
	c	Parhypate hypaton	fa	ut		
	H	Hypate hypaton	mi			
	A	Proslambanomenos	re			
	Γ		ut			

SYNEMMENON SYSTEM

Tetrachord		*Greek Name*	*Hexachords*		
	d'	Nete synemmenon			la
Synemmenon	*c'*	Paranete synemmenon			sol
	b	Trite synemmenon			fa
	a	Mese		la	mi
Meson	*g*	Lichanos meson		sol	re
	f	Parhypate meson		fa	ut
	e	Hypate meson	la	mi	
Hypaton	*d*	Lichanos hypaton	sol	re	
	c	Parhypate hypaton	fa	ut	
	H	Hypate hypaton	mi		
	A	Proslambanomenos	re		
	Γ		ut		

dieses and limmas are not sung in succession, like tones and semitones; and for that reason I did not want to call them minor semitones, as some do. For that reason, therefore, there had to be set at the beginning of the lines and intervals, which are the images of the strings, certain symbols by which to distinguish hard melody from soft, and to establish a place for the semitone, whether it be ordinary or even one of the others split off from major tones. Also there is another more hidden necessity, not from the difference of the kinds, but in any melody, from the inequality of the major and minor tone, to make them distinct from each other on the staves. For the syllables *Ut, Re, Mi* must be general and applicable both to the major and to the minor tone, so that those designations are always satisfactory for those learning to sing, and for practical instrumentalists; but for theoreticians they are most unsatisfactory. From this function, therefore, when they are set at the beginning of staves, these letters are generally called clefs [in Latin, keys], because without them the entry to the stave is not clear to the singer.

However, the same letters are also used for musical instruments; for they are inscribed on the levers of the plectra in the case of stringed instruments, or of the stops in the case of wind instruments; and from that inscription the levers are called in a special sense "keys," the actual arrangement, or the system of all the levers, the keyboard, *das clavier*, and the kind of instrument a clavichord, because the strings are struck by keys, that is by levers pressed down outside, and by plectra springing up inside.

The delineation of staves.

Furthermore there are staves, generally written for instrumentalists, on which instead of black lines there are series of letters, indicating which keys are to be touched. In that case the letters are instead of the symbols which the staves for singing used, placed on the lines or in the intervals.

About these letters, some happy observations occur to us. First, not all the letters are placed on their lines and intervals, but only *F*, *G*, and *C* whenever one of them is to be placed on a line, and *B* also in the case where the place for it is in an interval.

Origin of the symbol 𝄡.

Next, the letter *C* has a symbol which is very different, that is to say the one shown in the marginal note. I believe this symbol was born from a distortion of the ancient way of writing the letter *C*. For because the scribes used pens with broad points, many notes in abbreviated writing became square; and *C* could not be drawn as round with those pens. Thus they made *C* with three little lines, one thin, in place of the hollow part, and two thick, in place of the horns, using the width of the pen like this: ⊏. Now the thin line, as the pen hurried, very often became longer, and sticking out at both ends beyond the horns, like this: ⊨. However so as to finish off the horns, they drew little lines parallel to the first, like this: ⫤. And thus in the end two such lines became one, like the first and parallel to it, and the whole symbol like this , which, like the notes themselves on the lines, by the split in the pen is made hollow, like this .

The question also arises whether this symbol was not born from vocal practice, as it presents the shape of a ladder, as if it marked the beginning of the musical scale [in Latin, ladder] where it was placed. Or is it more true, that a picture of a ladder was first made from the letter *C*, and afterwards from the frequent use of it what was properly speaking a stave began to be called a scale?

On the letter *f*.

Now when the position of the letter *C* is not on a line (because at one time there were usually only four lines), then the position of the letter *F* is on one of the lines. It is written in the case of the other higher notes by its true symbol, in the bass by a semicircle curving back on itself, with two dots nearby. I think this symbol at the beginning, before it was distorted, was a minuscule Greek γ; and thus the two dots added to it signify digamma, because our *F* is the Greeks' double Γ.[99] For Vincenzo Galilei argues convincingly that this method of representing the notes of the system by the letters of the alphabet has come down from the Greeks, producing very ancient examples of Greek songs.[100] And it is admitted that even the simple Greek has found a place in our Latin music, as will now be told.

This *F*, therefore, is the symbol for another clef; and both symbols[101] are used next to each other, where appropriate places for each occur. The third is *G*, which is expressed in the case of higher notes, when there is a place for it on the line. The symbol also occurs in the ancient style on the lowest line of the stave. For the musical scale today begins with the note Γ*ut*, or *Gammut*, which indicates the lowest sound *G*. Thus Digamma and Gamma are the same as *F* and *G*.

On the letter *g*.

[102]But *b* also is drawn on the stave for soft melody in addition, to distinguish the kind of melody, whether it belongs on the line, or in the interval: its presence signifies soft melody, its absence hard. But not only that: it is also found regardless on other lines or intervals, which are not allocated to the key *b*, and by a kind of abuse the letter *b* is repeated instead of the symbol for a semitone or for the syllable *Fa*.

On the letter *b*.

On the other hand when exceptionally a semitone is intended instead of a tone,[103] and the syllable *Mi* instead of the syllable *Fa*, then

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[99] In the Greek alphabetical numerical system, taken over from the Phoenicians, digamma was the symbol for the number 6. See Menninger (1969), 270. Taking Γ*ut* as the lowest note *G*, and assuming the Greeks used the alphabetical system for musical notes, digamma thus represents the sixth note *F* above.

[100] Galilei's examples (*Dialogo*, 96–97) of four songs in the Lydian mode, with the melody expressed in the form of alphabetical symbols, were transcribed from a Greek manuscript which had been found in the library of a Roman Cardinal.

[101] That is, *C* and *F*.

[102] The letter *b* is used by Kepler to indicate that the note makes a minor third or minor sixth with the fundamental. It thus denotes strictly a note lowered by a diesis, or as Kepler points out, a note lowered by a semitone.

[103] Kepler seems to have meant, "when a tone is intended instead of a semitone." As noted already, Kepler used the symbol both to cancel a flat and to indicate a sharp.

On the letter ♮.

in front of the note is put the letter ♮ or a symbol derived from it. For the ancients beyond doubt drew it like this ⊟; but instead of that we use ✱ or ✱ which Galilei considers to mean the same to the reader as the Greek word diaschisma (split) once did.[104] For it is plain enough that it denotes a cleft, and indicates to us the splitting off of semitones, which we have dealt with in the previous chapter, VIII.

On the keys of keyboard instruments[105] we do not perpetrate this ambiguity with the letter *b*, that is to say in a position which is not its own. For in the first place all the keys or levers for those notes which are formed by splitting off semitones from a tone, and which increase the eightfold number of the octave system, are both projecting, or standing up from the main level keys, and black in color. Secondly, on each black key is written the letter for the neighboring key, but with a tail, to distinguish it (except for ♮ in the case of which a plain *b* is put on the black key): and the tail signifies a higher note, that is *f*, *f*ϱ, *g*, *g*ϱ, *b*, ♮, *c*, *c*ϱ, *d*, *d*ϱ, but between *f* and *f*ϱ, *g* and *g*ϱ, *c* and *c*ϱ there is a limma, and a diesis between *b* and ♮, that is to say between the same letters for which also two notes are shown in only one position on the stave. But between *d* and *d*ϱ there is a semitone; just as these two letters, although they are the same, are nevertheless not written in the same position, but in successive positions.[106] However, there is a diesis between *d*ϱ and *e*, and these two different letters belong in the same position. This difference is a concession which has to be made to the ancient custom, and is mitigated in practice. Finally between *f*ϱ and *g*, *g*ϱ and *a*, *a* and *b*, ♮ and *c*, *d* and *d*ϱ, and *e* and *f* are semitones. Thus caution must be used here so that by some analogy from their being written in the same way we do not make any assumption about the identity of the interval.

It is also especially remarkable, and must similarly be conceded to ancient custom, that in many clavichords and organs (in which the octaves are in continuous succession), although on account of the identical consonance of the diapason the same letters are quite justifiably repeated, yet that is not done starting from the first *A* but from the second *b*. For this is the order of the characters: *C*, *D*, *E*, *F*, *G*, *A*, *b*, *c*, *d*, *e*, *f*, *g*, *a*, *bb*, *cc*, *dd*, *ee*, *ff*, *gg*, *aa*, *bbb*, *ccc*, *ddd*, *eee*, *fff*, *ggg*, *aaa*, etc.

The system.

Further, what a stave is on paper, a system is on an instrument, that is to say a series of all the strings dividing one consonant interval; and as was stated in Chapter V, first of all a diapason is convenient for the interval; then for all those larger intervals which are of a size which any instrument embraces.

Although it is not part of my intention to dwell much on the music of the ancients, as it is full of obscurity, yet I have not thought it right to pass over the names by which they invoked the eight notes in a sys-

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[104] Galilei, *Dialogo*, 6–7.

[105] Keyboard instruments is intended.

[106] See Chapter 8 above.

tem of one octave; for that designation has a relationship with the letters with which we have dealt hitherto.

The antique names of the strings.

The ancients, then, in the time of Aristotle[107] enumerated the strings in the following way: Hypate [highest], Parhypate [next to highest], Lichanos [forefinger], Mese [middle], Paramese [next to middle], Trite [third], Paranete [next to last], Nete [last]. The note of the Hypate was the lowest, and it seems to have been called by that name from its position on the stringed instrument, when it is in position for playing; for it takes the same position even today in the lute, bandura, and zither.

And since Aristotle asserts that between the Hypate and the Parhypate there was a diesis (that is, Plato's semitone, which was the remainder after subtracting two major tones from the diatessaron, 243:256, which he called a diesis) therefore the Hypate was the same as we represent by the letter *A*, and the parhypate as the letter *b* represents. Therefore the Lichanos (so called from the index finger) is *c*. Then the Mese is *d*; and it was called by that name because it was the middle one of seven strings, at the time when there were only seven strings on the psaltery, and the first string was struck instead of the octave.[108] The Paramese has the letter *dϱ* or *e* (this the more ancient omitted),

[107] The *Problems* of Aristotle, which Kepler evidently used as one of his sources, was not in fact written by Aristotle himself but probably by some of his students. For a discussion of some of the problems of Pseudo-Aristotle concerning music, see Barker (1984), pp. 190–204. A clear account of the building up of the octave in early Greek music is given in Boethius, *De institutione harmonica*, Book I, Chapter 20. The earliest system consisted of four strings, spanning an octave, in which the inner strings both made a diapente and a diatessaron with the outer strings. This was expanded into a system of seven strings. Starting from the lowest, these were called Hypate, Parhypate, Lichanos, Mese, Paramese or Trite, Paranete, Nete. This was a synemmenon (connected) system, consisting of two tetrachords, Hypate to Mese and Mese to Nete. As the system did not span an octave, the first string, Hypate, was struck instead of the missing string an octave higher. An eighth string was then added by separating Paramese and Trite, to form a diezeugmenon (disconnected) system, spanning an octave, the two tetrachords, Hypate to Mese and Paramese to Nete, being separated by a tone.

[108] Kepler's interpretation of the synemmenon and diezeugmenon systems of the octave are illustrated in the following table. The Greek names are correct only in the case of the diezeugmenon system, in which there was indeed a diapente between Nete and Mese, as Kepler reports. In the earlier seven string synemmenon system, however, the interval between Nete and Mese was a diatessaron.

Diezeugmenon System	*Synemmenon System*	*Greek Name (according to Kepler)*
a	*a*	Nete
g	*g*	Paranete
f	*f*	Trite
e	*dϱ*	Paramese
d	*d*	Mese
c	*c*	Lichanos
b	*b*	Parhypate
A	*A*	Hypate

the Trite the letter *f*. It was called that because it was the third in order from the highest, just as we call the sixth from the lowest *A*. The Paranete is represented by the letter *g*; and the Nete, as the one which is touched last by the plectrum when it is drawn downwards, returns to the letter *a*. This distribution is also confirmed by the fact that there is said to have been a diapente between the Nete and the Mese.

However, such a system of eight strings could not be used for both kinds of melody unless either the Parhypate *b* and the Paramese *dϱ* were shortened by stopping with a finger so that they became ♮ and *e*, or they were tightened by the pegs in hard melody, or loosened in soft. And from that circumstance was born the differentiation of the sounds into fixed and movable, and the actual designation διέσις (diesis) from the slackening of the movable strings.

What were the fixed sounds for the ancients and which were movable?

In the combination of several systems of diapasons, the reasoning of the ancients was more intricate. However, it must be observed that they took in one string lower than the actual Hypate, which from that fact was called the Proslambanomenos (additional).[109] A trace of that survives in the musical scale, which our youngsters learn. It takes in below the Latin letter *A* one which signifies a lower note,[110] for it is represented as an extra by the Greek letter Γ for the scale begins like this: *Gammut, Are*, and so on.

The proslambanomenos.

Let it be enough to have touched on these aspects of the ancient names of the strings at this point. For below in Chapter XI all the fifteen strings of the combined system will be explained in the old terms, though it is that of later musicians.

Now let us at last proceed to our natural system for a single octave, and let us apply to it the letters from the normal music.

I therefore decide that in this natural and logical system the lowest note is the one which the ancients used to name the proslambanomenos, but the musical scale usually names *Gammut*, and of course that the letter to be allocated to it is *G*. The reasons are that the notes established by the harmonic sections mark out in fact below the octave those intervals which musicians ordinarily establish between *G* and *b*, ♮, *c*, *d*, *dϱ*, and *e*.

Natural system to start from *G*.

Someone may say that the same intervals may be discerned between *D* and *F*, *Fϱ*, *G*, *A*, *b* and ♮, and similarly also between *c* and *dϱ*, *e*, *f*, *g*, *gϱ*, and *a*. Nor do I deny that instrumental musicians generally substitute these letters, and do not distinguish in them between major and minor tones. For what is to prevent them transferring a melody from *G* to *d*, when on their instruments they conflate the two tones, the major 8:9 and the minor 9:10, into a single interval 4:5 and afterwards precisely bisect it into two conventional tones, which are equal

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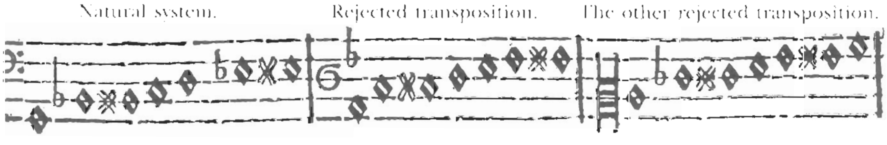
[109] This string, one whole-tone below Hypate, was introduced to keep Mese exactly in the middle. See Boethius, *De institutione harmonica*, Book 1, Chapter 20.

[110] Kepler is mistaken in supposing that the note Γ introduced by Guido d'Arezzo is identical to the Greek Proslambanomenos. Γ is a tone lower. See note 98.

to each other?[111] Though Galilei is more accurate in this tempering.[112] However, we are looking here not at the ἀτεχνία [deficient technique] of the empirical, but at the ἀκρίβεια [accuracy] of nature; and so we cannot imitate the former. And even they have in fact rejected the letter *c* as the source of the natural system of the octave, by the following argument.[113] For between the notes of *gϱ* and *a* there is a semitone: and this interval is in the fifth place, as *a* is the sixth from *c*. But in the natural system we need a diesis before the sixth place. However the musicians themselves place a diesis only between *b* and ♮, and between *dϱ* and *e*. Therefore with the agreement even of the actual instrumentalists our natural system must stem either from *D* or from *G*, for a pure natural system to be expressed.

But a natural system cannot start from *D* either. For it is their normal practice to place a semitone between *d* and *dϱ*, whereas in my natural system there is a limma in the lowest position, which is smaller than a semitone.

Natural system. Rejected transposition. The other rejected transposition.



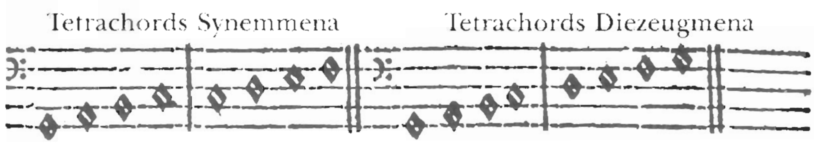
[111] Kepler's knowledge of this temperament goes back to a letter from Seth Calvisius (KGW 16, p. 56).

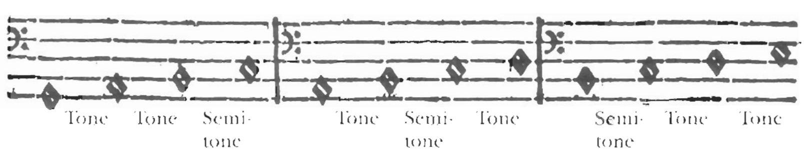
[112] Galilei used the value 17:18 for the tempered semitone, applied in practice for the spacing of the frets of the lute and viol. Kepler had himself used this value in calculating the twelve semitones of the tempered scale. See notes 92 and 93.

[113] The intervals for Kepler's natural system are set out in Chapter 8. Kepler has shown that, if the instruments are tuned to these natural intervals, then the scale must start on *G* and a melody cannot be transposed to a scale beginning on *D* or *C*.

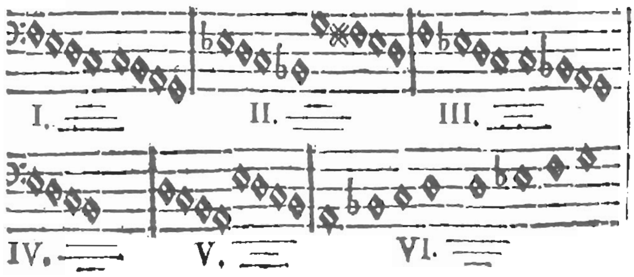
Chapter X.

On the Tetrachords and the Use of the Syllables *ut, re, mi, fa, sol, la.*

It was conventional for the ancients to separate the system of the octave into two tetrachords, and that in different ways according to the different intentions of the musicians. For they were either joined together, and called synnemena [connected], with one note fixed in between, which produced a diatessaron both with the lowest note below and with the highest note above. Thus the extremes made a dissonant interval, the diahepta [seventh], and it was tacitly understood that there was one additional tone above. For whenever the arrangement demanded the striking of the octave *a*, they struck the first *A*, as if it were giving out a note which was identical by opposition, or "antiphonal." On that basis there were not truly two tetrachords, but in fact two intervals of diatessaron, but also one heptachord. Or else the tetrachords were separate, with an interval of one major tone, and called Diezeugmena [disconnected], which the nature of things also persuades us applies in our own division of the system, in which the lower tetrachord contains *G, A,* ♮, and *c*, and the upper *d, e, f*♯ and *g*, in which case there is a major tone between *c* and *d*. Perhaps the four lowest strings were also separated by their actual position from the four upper ones, leaving, that is to say, a major gap between the two on the boundaries.

The reason for their deliberating about the system of the tetrachord was that they saw that in the one consonance of diatessaron there were two tones and a half; and we have a major tone, a minor tone and a semitone in the perfect diatessaron, that is to say in a minor consonance (for 4:5 and 5:6 were not even considered as consonances), all melodic elements, and a semitone either in the lowest or the middle or the highest position. Therefore all melody seemed to them to be included in the three forms of tetrachords.



For me the forms of tetrachords are multiplied, because of the distinction between minor and major tones. For instead of three, six occur, and in one diapason several double forms. Thus we have a kind of franchise from nature for differentiating the tetrachords, as the upper one is in a way like the lower one in the disposition of the elementary intervals.

A semitone is signified by a short line, a minor tone by a medium line, a major tone by a longer line.

		Below	In middle	Above
First form in	*c d e f* / *G A* ♮ *c*	Major tone,	minor,	semitone.
Second form in	*b c d dǫ* / *d e fǫ g*	Minor tone,	major,	semitone.
Third form in	*c d dǫ f* / *G A b c*	Major tone,	semitone,	minor tone.
Fourth form in	♮ *c d e*	Semitone,	major tone,	minor tone.
Fifth form in	*A* ♮ *c d* / *d e f g*	Minor tone,	semitone,	major tone.
Sixth form in	*A b c d* / *d dǫ f g*	Semitone,	minor tone,	major tone.

And these are only the forms of the perfect distessaron: with the imperfect ones we shall deal in Chapter XII.

However if we disguise the difference between a major and a minor tone, as the instrumentalists do, then the separated tetrachords are in all respects similar, in succession as follows.

G A ♮ *c*	and *d e fǫ g*
G A b c	and *d e f g*
This is the lowest	and this the highest.

In each case there is a semitone in the same place, either the highest or the middle, with the sole difference that the semitone in the lower tetrachord is higher by one comma than the one in the higher. This difference is not much noticed in performance.

On this similarity of the tetrachords, therefore, depends the transposition of melody from key to key, which is conventional for musicians. It is the more convenient because the instrumentalists have removed the distinction between major and minor tones, and so forth. Transposition.

Again this similarity and triple variety of the tetrachords has given birth in more recent music to the six syllables *ut, re, mi, fa, sol, la*, which

Use of the syllables *Ut, Re, Mi, Fa, Sol, La.*

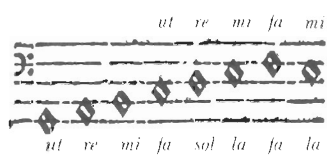
assist the memory of learners. But if only one octave were sung, made up of two similar tetrachords, we could be content with only four syllables, *ut, re, mi, fa, re, mi, fa,* disguising the distinction between the major and minor tones.

But because there are three positions with a semitone in the tetrachord, then so that these syllables should not be too general, but rather a semitone should always be denoted by *Mi Fa,* or *Fa Mi,* two other syllables had to be added, so that in the cases of *Ut, re, mi, fa* there should be a semitone in the highest position, but in the cases of *Re, mi, fa, sol* a semitone in the middle position, and lastly in the cases of *Mi, fa, sol, la* a semitone in the lowest position. And that is the reason why the musical inventors used six syllables, not eight. Then let the Belgian who made seven instead of six, *Bo, ce, di, ga, lo, ma, ni,* consider what good this increase does him. For if he supposed that sounds must be substituted which were equal in number to the strings of one octave, less one, so that the octave to show its identity should be signified by the first syllable *Bo,* what, pray, does he find wrong with the letters *a, b, c, d, e, f, g,* which have long been accepted for this purpose?

Modulation of melody.

Since therefore it was in this way that the six syllables were born, on account of the threefold form of the tetrachord, from that the theory of modulations was born, through the modern musical scale. That theory is concerned only with these syllables, which are to be fitted to the letters explained above, and shows that the superfluous syllables *Sol, la,* are sometimes equivalent to the first, *ut, re,* and sometimes to the succeeding ones, *Re, mi,* so that sometimes *fa* can be imported outside the series and without modulation after *sol, la.*

1.	2.	3.	4.	5.	6.	7.	6.
Ut,	*re,*	*mi,*	*fa,*	*sol,*	*la,*	*fa,*	*la.*
Ut,	*re,*	*mi,*	*ut,*	*re,*	*mi,*	*fa,*	*mi.*



However there is also another dissimilar similarity in the tetrachords, which is the basis of the blending of different sequences of notes which are in tune with each other, which they usually call fugues or fantasies. For if you take neighboring tetrachords not separated from each other by any interval, in such a way that there remains a major tone to complete the octave above or below, there will arise in certain systems of the octave, on each side, the same sequence of elementary intervals, that is of semitones, dieses and limmas. One is supplemented by a major tone, so that instead of a fourth a fifth is produced, in a way similar to the other fourth; and the two tunes, the one earlier

Semitone, Limma } Additional tone

Similar tetrachords:

Higher:
- Semitone, Diesis } Minor tone
- Semitone
- Semitone, Limma } Major tone

Lower:
- Semitone, Diesis } Minor tone
- Semitone
- Semitone, Limma } Major tone

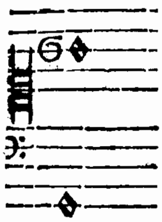
in time running over an interval of diapente above, the other later in time running over an interval of diatessaron below, follow each other, the later after the earlier, and as far as it may imitating its outline as if in rivalry to it.[114]

[114] Kepler's example in musical notation does not correspond to the table of intervals. This table corresponds to an example in which the lower tetrachord goes from *g* to *cc* and the upper from *cc* to *ff*, so that the additional note is *gg*. Cf., the table in Chapter 11.

CHAPTER XI.

On the Combination of Systems.

Although one man's voice is often encompassed within the system of a single octave, yet the singing of several lines of melody together teaches us how to arrange several neighboring octaves. Therefore the ancients, who defined everything by the properties of numbers, defined the perfect system of the diapason epidiapente [the fifth above the octave] by the numbers 1, 2, and 3. We found above that the individual harmonic divisions yielded individual systems, of which the largest was the disdiapasonepidiapente [the fifth two octaves above], with the numbers 1, 5, and 6. However for our polyphonists as they call them, no limit is set when they multiply the number of lines of melody to be sung together, as God the Creator also, in his tempering of the heavenly motions, preceded them, making a system of seven diapasons, and more. Nevertheless because here we are putting together a system for examining absolutely all the consonances, perfect and imperfect, for that task two diapasons put together in one interval, in the following way, are sufficient, and Ptolemy also called it a perfect system.[115]



It is not my intention to deal with the other parts of the music of today, for it has nothing to do with the nature of the intervals, which is the measure of the notes, their variety, the proportion of the modes, the pauses, and the like.

[115] See Ptolemy, *Harmonica*, Book II, Chapter 5. Ptolemy distinguished two ways of naming the notes, which he called *thesis* (position) and *dynamis* (function). In the positional system, the lowest note is Proslambanomenos, regardless of the sequence of intervals. An example of this positional system is here illustrated (perhaps unintentionally) by Kepler. In Ptolemy's functional system, the note immediately below the higher disjunction is always Mese, wherever it falls in the double-octave space. Thus functional Mese is always *a* and functional Proslambanomenos is always *A*. Cf. note 98. Kepler probably believed that, in assigning Greek names to the notes, he was describing Ptolemy's functional system. The confusion arises from his mistaken belief that Guido d'Arezzo's hexachordal system identifies Γ*ut* with the Greek Proslambanomenos. As already noted, Γ*ut* was a tone lower than the Greek Proslambanomenos.

Our musical scale, which is nothing other than the largest system of our music, embraces two diapasons and one sixth. I append it here, in comparison with the system of the ancients, and the Greek designations of the strings.

Strings with keys and intervals within the double octave.

String	Interval	Number
gg		540
	Semitone	
*ff*ϱ		576
	Limma	
ff		607½
	Semitone	
ee		648
	Diesis	
*dd*ϱ		675
	Semitone	
dd		720
	Semitone	
*cc*ϱ		768
	Limma	
cc		810
	Semitone	
𝔥𝔥		864
	Diesis	
bb		900
	Semitone	
a		960
	Semitone	
*g*ϱ		1024
	Limma	
g		1080
	Semitone	
*f*ϱ		1152
	Limma	
f		1215
	Semitone	
e		1296
	Diesis	
*d*ϱ		1350
	Semitone	
d		1440
	Semitone	
*c*ϱ		1536
	Limma	
c		1620
	Semitone	
𝔥		1728
	Diesis	
b		1800
	Semitone	
A		1920
	Semitone	
*G*ϱ		2048
	Limma	
G		2160

Here the individual lines signify individual principal strings.

String	Syllables	Greek designations of the strings from Ptolemy's perfect system.
ee	*la*	
dd	*la sol*	
cc	*sol fa*	
bb	*fa* ♮ *mi*	
aa	*la mi re*	
g	*sol re ut*	Outermost Nete
f	*fa ut*	Outermost Paranete
e	*la mi*	Outermost Trite
d	*la sol re*	Disconnected Nete
c	*sol fa ut*	Disconnected Paranete
b	*fa* ♮ *mi*	Disconnected Trite
a	*la mi re*	Paramese
G	*sol re ut*	Mese
F	*fa ut*	Middle Lichanos
E	*la mi*	Middle Parhypate
D	*sol re*	Middle Hypate
C	*fa ut*	Uppermost Lichanos
B	*mi*	Uppermost Parhypate
A	*re*	Uppermost Hypate
Γ	*ut*	Proslambanomenos

Here not only the lines but also the individual intervals between the lines signify individual principal strings or notes, as in the staves of today.

Arithmetic and mechanical corollaries

Multiplication of the length of the string *gggg* produces eleven strings.

Let *gggg* be	135
	135
ggg	270
	135
ccc	405
	135
gg	540
	135
*dd*ϱ	675
	135
cc	810
	135
—	945
	135
g	1080
	135
f	1215
	135
*d*ϱ	1350
	135
—	1485
	135
c	1620
	135
—	1755
	135
—	1890
	135
—	2025
	135
G	2160

And here *d*ϱ and *f* are a sixteenth part of the string apart,[116] in which there is a minor tone, and *f* and *g* by a major tone.

Multiplication of the string *cccc*ϱ gives rise to eight strings.

Let *cccc*ϱ be	192
	192
*ccc*ϱ	384
	192
*ff*ϱ	576
	192
*cc*ϱ	768
	192
—	960
	192
*f*ϱ	1152
	192
—	1344
	192
*c*ϱ	1536
	192
𝔥	1728
	192
A	1920

Here also A and 𝔥, and 𝔥 and *c*ϱ are one ninth apart, not of the whole string but of 𝔥 or one tenth of A.

Or divide G into 5 parts, one of those into 3, and that again into 3, so that 45 parts are made.
Of these 12 are *ff*ϱ, 15 are *dd*, 16 are *cc*ϱ, 18 are 𝔥𝔥, 20 are *a*, 27 are *e*; accordingly 24 are *f*ϱ, 30 are *d*, 32 are *c*ϱ, 36 are 𝔥, 40 are A.

Multiplication of the string *fff*ϱ gives rise to six strings.

Let *fff*ϱ be	288
	288
*ff*ϱ	576
	288
𝔥𝔥	864
	288
*f*ϱ	1152
	288
d	1440
	288
𝔥	1728

Here also *d* and *f*ϱ are a sixth part of 𝔥 apart.

Another division of the monochord. Divide *G* into three parts, and any one of those into 2, 4, 8, and 16, so that 48 parts are made. Of those 12 are *gg*, 15 are *dd*ϱ, 16 are *dd*, 18 are *cc*, 20 are *bb*, 24 are *g*, 27 are *f*, 30 are *d*ϱ, 32 are *d*.

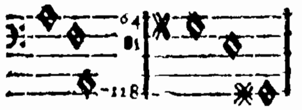
Again divide *G* into 5 parts; of these 3 will be *e*, 4, 𝔥. Lastly divide 𝔥 into 3 parts, and each of those into 3, so that 9 parts are made. Of these 8 give *c*ϱ, 10 give A. Lastly trisect *c*ϱ, and take 4 of these parts for *g*ϱ.[117]

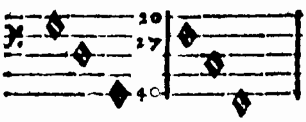
[116] That is, the *G* string.
[117] This should be "trisect *c*ϱ and take 4 of these parts for *G*ϱ."

Chapter XII.

On the Impure Consonances.

From the continuation of two diapasons there result imperfect consonances. For in joining the two together there follow in succession two major tones between *f* and *g*, and *g* and *a*. Thus in the case of *f* and *a* the interval is 64:81, which is greater by one comma, 80:81, than a just major third, 4:5 or 64:80. Let us call it a ditone [double tone], with Galilei.[118] There is the same interval between *e* and *g*ϱ, because between *e* and *f* and *g*ϱ and *a* on each side there is the same interval of a semitone. The Ditone





Thus on adding a semitone to the oversize ditone between *f* and *a*, the interval between *f* and *bb* becomes 60:81 or 20:27, which is greater by a comma, 80:81, than one fourth 3:4 or 60:80.

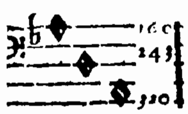
Make the same conclusion about *e* and *a*, because just as *e* is a semitone less than *f*, so *a* is than *b*.[119]

If these imperfect or oversize consonances are subtracted from the interval of a diapason, they leave remainders which are imperfect and undersize by one comma. For between *a* and *ff* in the same octave will be 81:128, an undersize minor sixth.

There is the same between *g*ϱ and *ee* if from the octave you subtract the interval *e–g*ϱ.

In the same way between *bb* and *ff* in the same octave, after subtraction of the imperfect fourth *f–bb*, 20:27, from the octave, the remainder will be a narrow fifth,[120] 27:40, that is 81:120.

There is the same between *a* and *ee*, on subtracting *e–a* from the octave.



However by adding the oversize ditone *f–a* to the minor third *a–c*,[121] 5:6, is begotten 160:243, a fifth or diapente *f–cc*, which is oversize by one comma.[122] For 160:240 is 2:3, and 240:243 is 80:81.

And on taking away the oversize fifth *f–cc* from the interval of an octave, there remains a fourth *c–f*, 243:320, which is undersize by one comma.

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[118] Galilei, *Dialogo*, pp. 12 and 32.

[119] This should be *bb*.

[120] This has an application to the cosmic harmony. See Book V, Chapter 9, proposition 43.

[121] This should be *a–cc*.

[122] Here Kepler has made a mistake, which is the consequence of his failure to recognize that the minor third *a–cc* is one comma undersize. It follows that the fifth *f–cc* and also the fourth *c–f* are pure.

Table of the Six Imperfect Consonances.

For 4:5 there is 64:81, oversize
For 3:4 there is 60:81, oversize, or 20:27 and 243:420, undersize
For 5:8 there is 81:128, undersize
For 2:3 there is 81:120, undersize, or 27:40 and 160:243, oversize.

However there is no need for the Platonic semitone or limma,[123] 243:256, because he called it a limma for a different reason from me. For on taking away two major tones, that is to say a ditone, 64:81, from the interval 3:4 he found the remainder was this limma, 243:256, in place of a semitone. But I have no need for that subtraction, for I prefer the interval 3:4 to be a wide fourth, and it then becomes 20:27.[124]

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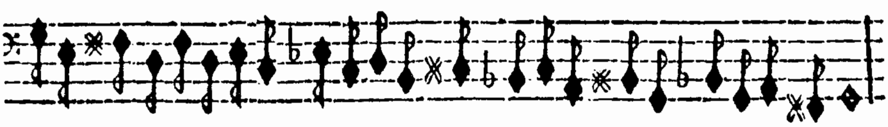
[123] It should be noted that Kepler's limma (128:135) is different from the Platonic semitone or limma (243:256). The division of the tetrachord into a Platonic limma and two major tones was used by Ptolemy in the species of the diatonic scale that he called "diatonon ditoniaion" (ditonic diatonic). *Harmonica*, Book II, Chapter 14.

[124] The system described by Kepler in the last two chapters is in effect the system of Ptolemy, which uses pure natural intervals. Although the elimination of the impure consonances was an unattainable ideal, Kepler's system includes only a small number, some too narrow and some too wide by a comma. For a table showing the locations of the impure consonances in Kepler's system, see Dickreiter (1973), 157. It should be noted, however, that the signs in this table are incorrect in the cases of the minor thirds and major sixths. The minor thirds are in fact too narrow and the major sixths too wide by a comma. For Kepler the system was above all a theoretical system, essential for a knowledge of the nature of music. He recognized that, in the practice of music, and especially in the playing of instruments, a tempered scale was generally employed.

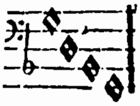
Chapter XIII.

What Naturally Tuneful and Suitable Melody Is.

We shall say nothing of that grating style of song which the Turks and Hungarians customarily use as their signal for battle, imitating the uncouth voices of brute beasts rather than human nature.

It is quite clear that the original author absorbed uncouth melody of this kind from an instrument which was rather unsuitably shaped, and from long familiarity with the construction of the instrument transmitted such melody to his descendants and to his whole nation. I have been present in Prague at the prayers which the priest of the Turkish ambassador used to chant at fixed hours, kneeling and frequently hitting the ground with his forehead. It was easily seen that he sang from training, and had labored to acquire a practiced and fluent manner, for he did not hesitate at all; but he used remarkable, unusual, truncated, abhorrent intervals, so that it seems that nobody could with proper guidance from nature and voluntarily of his own accord ever regularly contemplate anything like it. I shall try to express something close to it by our musical notation.

Therefore melody which is tuneful and suitable by the judgement of human ears starts from some particular note, progresses from it by melodic intervals to notes which are consonant both with the first one and generally also with each other, flitting hastily over dissonant intervals, but dwelling on consonant ones, whether by the measure of time and the length of syllables, or by frequent return to them, as if aspiring to the consonance of two parts with each other, by the passage of one voice from one place in the system to another. Example:

[The Easter victim let Christians offer praises. The Lamb redeems his sheep. Christ the innocent has reconciled sinners to the Father.]

Here the initial note is on the note *G*, with which in soft melody *c*, *d* and *g* are in consonance. The melody therefore hurries (first making a downward turn) to the note *c*, which is consonant, and leaps over the plainly dissonant position *A*. However it would have been the same if it had touched on it, but lingering briefly. However the whole of the rest of the sequence rings out chiefly on the positions of *b*, *d* and *g*, exhibiting them as the skeleton of the octave, most frequently

returning to *d*, and next to it *b*, but from time to time reaching up to *g* above, and to all those positions significantly, but not in that way to *a* or to *f*, positions which are primarily dissonant; and at length it returns to *G* and ends there.

About the accepted definition of melody many points occur to us as calling for comment.

I. As components of melody, of all or some of which all melody consists, Euclid[125] names these four: 'Αγωγὴ [approach], Τονὴ [emphasis], Πεττεία [gaming], Πλοκὴ [twisting.]

What they are, the words themselves indicate. For 'Αγωγὴ is the passage of the voice from a stated point of origin to a position which is always consonant with the point of origin, or from one which is consonant to another which is consonant either with it or with the first point of origin. Τονὴ is dwelling either on the first position or on one consonant with it, or even consonant with an earlier position though not with the first. Πλοκὴ, entwining, is a species or color of ἀγωγὴ, as πεττεία, playing, is of τονὴ; and as 'Αγωγὴ is to Τονὴ, so πλοκὴ is to πεττεία; because ἀγωγὴ passes on, so to speak, directly, πλοκὴ wanders in its passage around the ἀγωγὴ, as a dog does around a passerby.

Thus ἀγωγὴ is like a direct motion, πλοκὴ like a variable motion; whereas τονὴ is, so to speak, the termination of motion, or rest in the position in the system which has been the aim, πεττεία is many terminations of rather tiny motions, like rests. In our example, the syllables "-demit oves Christus in-" are a sort of continuous τονὴ with the exception of two syllables; but the syllables "-cens . . . re- . . . li- . . . peccat-" exhibit a πεττεία. But if in this tune you inspect the tied syllables, "laud-, -an-, patr-, -tor-," and consider them as having two notes, then there will be no pure and simple ἀγωγὴ in it. But if you consider how it has come about in the style of folksong that from a note which was to be sung in a rather drawn out way has been made a double note which rises at the end, and if to these syllables you restore the simple drawn out note which is the first of the tie, you will find a pure ἀγωγὴ in "Paschali laudes," also in "immolent," also in "Christiani," also in "innocens patri," and a short one in "-catores"; but there is a pure πλοκὴ, although not a natural one, in the Turkish example.

Therefore as the skeleton is to the body for anatomists, so in a single system of an octave are the sounds which are consonant, both among themselves and with the point of origin or base of the octave,

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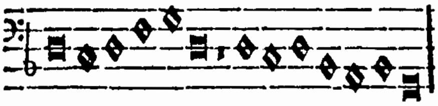
[125] *Euclidis opera omnia*, ed. J.L. Heiberg and H. Menge (Leipzig, 1883–1916), vol. 8, p. 222. *Introductio harmonica*. For a German translation, see O. Paul (1872), 230–244. It is now accepted that the author of the work is Cleonides, the student of Aristoxenus, to whom it is attributed in some manuscripts.

to the actual tune or melody. For just as flesh fills out the curves of the bones, and clothes them to make them comely, so the components which have been listed fill out the skeleton of the octave, especially ἀγωγή and πεττεία, and straying over the dissonant positions which are scattered among the consonant notes shape and give body, so to speak, to the tune.

II. Though the definition of melody is that it starts from a certain definite note, which is the base of the system of the octave, that is not to be understood as always applying in practice. For variety is often



delightful, and serves for emphasis. Very often a tune begins from another note or position as if broken off; but implicitly a definite basic starting point is understood to be specified, and shines out everywhere in the whole course of the tune, as in this very old German one[126] it is easily understood that although in practice it starts from *d* yet the point of origin is *G*.

III. There is a similar exception to the statement that regular melody with its ἀγωγή or πεττεία keeps to the limits of positions which are consonant with the point of origin of its octave. For frequently these components in the middle of the course of a rather long tune occupy positions which are dissonant with the first point of origin; but that is done for the sake of variety, and it is just as if with the former tune there were mingled a new tune, and some new starting point were fixed for it which was dissonant with the former, or the skeleton of a new octave were signified. It is like an interlude or digression in a speech; so we do not linger over such matters but revert quickly, so to speak, to the principal skeleton. And as long as πονή or πεττεία occupies dissonances, we understand that the melody is not yet finished; for at the actual true finish it must return not just to consonant notes, in fact regularly, but also to the actual point of origin of the proper octave. From this structuring of the melody, and striving for consonances, the ancients seem even to have called simple melody "harmony" [fastening] exactly as if the fitting and suitable correspondence of the limbs, which is the soul of form, were called harmony, or as it were beauty; for Harmony was also the name of a woman.

IV. As far as ἀγωγή is concerned, it again varies; for it does not always in practice pass through all the intervening positions, but often leaps over some, and by the leap is carried from one note to a note consonant with it. And in that case it is ἀγωγή only potentially, and almost pure πεττεία or τονή, which is the style most frequently used by our trumpeters, when they sound battle orders or set the pace for the cavalry; for then almost nothing is heard except the skeleton of Leaps.

[126] The tune is the twelfth-century German Easter Hymn "Christus ist erstanden."

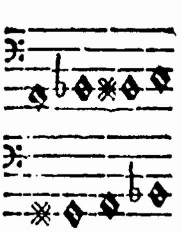
the octave. Ponder whether what the ancients knew as the Νόρος ὄρθιος [very high pitched "orthian strain"] was of this kind.[127]

V. Since tuneful melody consists of its elements, which are melodic intervals, they must first be defined more accurately.

The number of senses of unmelodic.

For indeed intervals are said to be melodic or unmelodic in two senses. For first the individual intervals separately are spoken of as melodic or unmelodic, the former in fact being those which nature admits in the divisions of the consonant intervals, but the latter those which do not result from comparisons of consonances, but are completely alien to harmony, as explained in Chapter IV. Secondly, in respect of the actual melody or tune, it is the distinction between intervals which are indeed in themselves musical, or in the first sense melodic, that certain of them considered in conjunction, although they are by nature melodic, that is to say they originate from the comparison and subtraction of harmonies, yet in practice are rejected from tunes in a certain respect as unmelodic. Τονὴ and πεττεία presuppose and take account of the former sense; but ἀγωγὴ, both authentic and intermittent, most of all the latter. Therefore they become unmelodic, again under two headings. For either it is in respect of the kind, either of the skeleton of the octave, because plainly they are not inserted into it, and do not shape it, or if they are arranged in any other pattern than nature has shown us (above in Chapter VII), as if someone wanted to arrange three semitones in succession. A semitone indeed is melodic in the first sense, but this arrangement of semitones is not melodic, in this second sense. Thus a diesis, which is by nature melodic, is yet either in the hard kind of melody or in the soft, separately unmelodic. Or else, finally, they are not performed in succession, nor permitted for a leap in the melodic line, nor suitable for ἀγωγὴ.

Therefore intervals which are unmelodic and forbidden in this second sense are included under the following laws particularly.



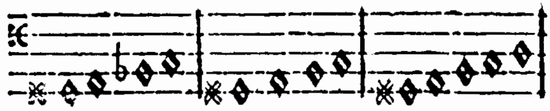
I. A diesis before or after a semitone is not performed as a rule.

II. Two semitones, although they are arranged in succession in the order of the smallest intervals in Chapter VII, yet cannot be performed by three successive notes, but must coalesce with two other elements into two tones.

III. Two semitones within the compass of a single fourth or fifth, which is in the lowest position in the skeleton of an octave, are not normally performed. For the effect would be that the note which was

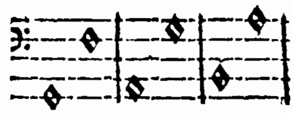
[127] The *nomoi* were types of early Greek solo compositions governed by strict rules. The Greek historians of a later time, when musical composition was more free, did not for the most part know what the rules were and in consequence the descriptions of the categories (such as *Orthios*), which have been transmitted to us, are somewhat vague. See Barker (1984), 249–255.

fourth or fifth in order would not be in consonance with the first, either perfectly or imperfectly, and thus it would not be truly a fourth or a fifth.

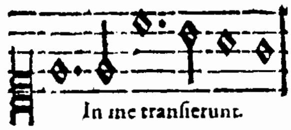


IV. Four tones are not performed in succession, except in the top part of the octave, when the skeleton is changed for the sake of color and variety; therefore not normally.

V. Sevenths, and all dissonances above the octave, with notes placed successively above the point of origin, are not normally performed, unless the preceding note provides something in the nature of an ending, and the following one something in the nature of a new beginning.

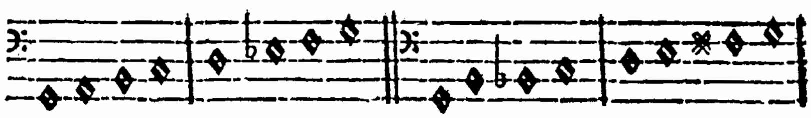


VI. We rather rarely admit sixths, although they are consonances, and only minor sixths.

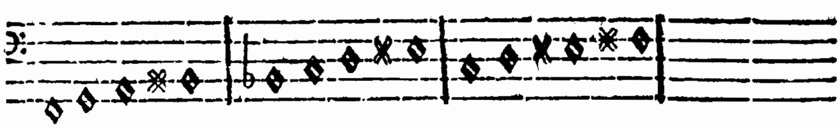


[They have come over to Me.]

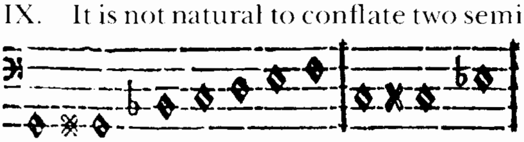
VII. The two tetrachords of a single octave, as there is not the same position for a semitone in both of them, are rather rarely performed, that is to say for the sake of color, or syncopations (which will be discussed in Chapter XV) and yet there is not complete freedom in placing the semitone, for it is restricted to the position which the kind of melody admits.



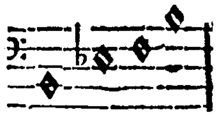
VIII. The fact that three tones in succession are not performed in the lowest position in the octave is related;[128] but they are admitted higher up by the laws already stated.



IX. It is not natural to conflate two semitones into one tone, although the difference to the hearing is small, as between *c* and *d* if the octave begins from *E*; for the interval would become 225:256 by the combination, which is naturally unmelodic, see Chapter IV.



[128] That is, to the reason given in VII.



X. In general every system of an octave which does not set up both a diatessaron and a diapente below is unmelodic.

Of these laws several will now be of service to us in establishing the number of the Tones in the following chapter.

VI. As far as the limb of the definition is concerned which states "flitting hastily over dissonant intervals, but dwelling on consonant ones," although it is not my intention in this work to present arguments on measures, however I mention in a general way that the pleasure of harmonic melody is chiefly in measure. On measures. For what has been said very generally, "flitting hastily over dissonant . . .," and so on, is made to come true in the particular case as follows. First, the Latin authors have a distinction between a long and a short syllable, and make the latter one unit of time, the former two, and thus of double length. In prose. Although today not only poetry but prose also is sung, in which the distinction between a long and a short syllable is completely neglected by the musicians of today, yet it takes into account solely the way of pronunciation, in spite of its being corrupt nowadays; and whatever is usually presented with a stress is taken as a long syllable, and whatever without stress, as a short.

It is not surprising that this happens in the Latin language, since the later Greeks have also done the same. Although they wrote correctly, yet they embraced an entirely faulty pronunciation, in which there is no longer any distinction between a stressed syllable and a long syllable, or an unstressed syllable and a short syllable, even though they have plenty of signs for that purpose, which among the Latin Greeks' verses popular. authors are not in such frequent use. Thus the Greeks have abrogated their style of composing poetry and began some centuries ago now to write verses which are called popular, in which the syllables are counted and not measured, and a verse has a penultimate syllable which In meters. is not long by nature but is stressed by accentuation in vulgar speech. That is therefore imitated even in Latin by our musicians, to such an extent that they do not even spare poetry, except very rarely, and never totally, I believe because the doublet and triplet are perpetually accepted in their songs, since poetry mingles these unequally, except for the hexameter, which is content with the ratio of equality in the spondee, and the perfect double in the dactyl. Therefore the musicians allot a short syllable, or one which is in place of a short syllable, when a spondee follows an iambus, and in general allot a syllable which is to be given with an accent, to a dissonant note, if they are to pass through it, and a long syllable to a consonant note, or nearly so.

Secondly, Ionic languages which use the Teutonic, French, Spanish, or Italian rhythm, make either the last or the next to last syllable of a poem stressed, and put out the whole verse just as if stressed or In today's meters. unstressed syllables follow each other alternately, whether that is appropriate, in Teutonic verse, or not, as often in French verses. Thus every verse of theirs is like either a Trochaic or an Iambic, and those either acatalectic, as "Schöpffer Himmels und der Erden" (Creator of the heaven and earth) or "Nun bitten wir den heylgen Geist" (now pray we to the Holy Ghost), or catalectic, as "Dess sich wundert alle Welt" (at that wonders all the world) or "Er ist der Morgensterne" (He is the morning star). Therefore the musicians fix the seat of dissonant notes in unstressed or shortened syllables, of consonant in stressed syllables.

Thirdly, as in fact melody takes its measure not from the syllables of the text, but from the beat, and that follows either double time or triple, they keep the first part of the measure in fact for consonances, but the later part, and in triple time the shorter, which is signified by a lifting of the hand, mostly for dissonances; and they take care as far as possible that the first half of the measure is only taken up by a syllable which is to be given out with a stress, or that long pairs of syllables do not intrude into the last part of the measure. In beat.

Chapter XIV.

On the Modes of Tunes Which They Call Tones.[129]

As I am about to speak of modes I first advise the musicians of today that I shall not discourse, except for what has already been said, on what they usually call modes, that is to say double time in the measure, triple time, and the like (modes such perhaps as they formerly made on the flutes in Roman comedies). By modes I mean those which they along with the ancients usually call tones, when someone asks what tone a melody is in. For there are certain qualities or species of tuneful melody, differentiated from each other into two main kinds, hard and soft.

The Greeks called them τρόποι ("fashions") and ἤθη ("manners"), the former from the pattern of the system, the latter from the effect of melody in man, because many legislators have resolved to make modes of melodies to control behavior.

Therefore about the number and distinguishing features of these tones not only are there great controversies among musicians today, but there have also been for a long time now, of which we may see some in Ptolemy, and many in Vincenzo Galilei.[130]

My plan however is not to examine nor to refute the opinions of everyone. Thus the musician is not to expect anything from me on this question but the following single point, to which my basic principles which have been explained so far lead me. For even though they seem to have been called tones from the fact that one mode starts higher or deeper than the next by an interval of one tone[131] (or semitone), yet I do not differentiate between the tones by the actual height or depth of the melody in itself, the only characteristic which Vincenzo Galilei testifies some to have observed[132]; but I demand, with Ptolemy, that the number of them should be in relation to the number of species of one diapason,[133] and I say that there are as many tones as there

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[129] The term τονοί was used by the Greeks to describe the modes. The medieval music theorists used the Latin version of this term, "toni," as a synonym for modes, which resemble the keys of modern music, and Kepler followed this practice. We have rendered Kepler's term as "tones." The term "tone" is also used of course to denote an interval. The sense in which the term is to be understood should be clear from the context.

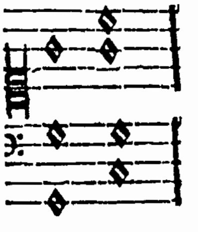
[130] On Ptolemy's treatment of the τονοί, see *Harmonica*, Book II, Chapters 7–10.

[131] This was Ptolemy's opinion. *Harmonica*, Book II, Chapter 10.

[132] The differences of the modes are explained by Galilei, *Dialogo*, 65–66.

[133] Ptolemy accepts seven modes, equal in number to the number of species of the octave. *Harmonica*, Book II, Chapter 9.

can be skeletons of the system of an octave which are legitimate and melodic, differing from each other in kinds and arrangement of melodic intervals, and in the position of the tetrachords, or in the selection of pairs of harmonic means. There are of course three things which make tones different and distinguish them from each other: kind, sequence of melodic intervals, and the construction of the skeleton from the minor consonances.

First the two kinds, established in Chapter VI, of division of the system of an octave, establish for us two chief tones, primary in nature, that is to say originating along with the system of one single octave. Secondly in each kind of system, both hard and soft, from the combination of two diapasons into one greater and perfect system, as we explained in Chapter XI, we have systems of individual octaves which differ in the points of origin or the boundaries within which they are confined. Now these boundaries are various positions in the natural system, or in the present day terminology different tonics, which are the necessary consequence of their differing pitch, in comparison with the natural system, which begins at *G*, and through that also of the differing position of the semitone, which is in the third or next to last or last position in the octave. For as far as the actual pitch is concerned, considered absolutely, it does not change in any way the kind of melody, since the same kind can be sounded at either high or low pitch. Nor are we confined to taking with Ptolemy half of the remaining tones as deeper than the chief and natural system of *G*, and half higher. For we can either take them all as higher, or all as lower, since the same arrangement of melodic intervals follows on both sides, provided of course the system of the naturally combined double octave begins at the *G* key. But if we make part lower, part higher, with Ptolemy,[134] we shall have to combine the natural system in such a way that we add a diapente below, a diatessaron above; and thus the combined system of a double octave will begin at *C*.

Lastly, the third point is that in systems established in that way we have a choice of two harmonic means, to be the common limits of τονή and πεττεία, as components of melody; or else, which comes to the same thing we have a position for the tetrachord either in the highest or the lowest place in the octave, or in the middle of the two thirds.

[134] Ptolemy constructs the modes by starting at different points on his chief scale and transferring notes from one end to the other, while keeping the intervals fixed, in order to build up the two-octave compass in ever case. If Kepler had wished to follow the same procedure, his addition of a fifth below and a fourth above the upper octave (as shown in his diagram), would have given him three modes below the natural, starting on *D*, *E*, *F*, and three above starting on *a*, ♮, *c*, so that his double octave would begin on *C*.

Therefore on account of these variations we must append here from Chapter XI, before everything, a square matrix fitted together from the twelve smallest natural intervals in the octave; not indeed so as to make with Aristoxenus[135] in accordance with the number of them the same number of tones, starting each of them higher by one such smallest interval, but so that with these smallest intervals assembled into the conventional melodic intervals it may be seen whether some new kind of octave can begin at any of the smallest intervals.

Matrix of the intervals in the systems,
for tracing the various kinds of tones.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX	X.	XI.	XII.	Arrangements
S	L	S	S	D	S	L	S	S	D	S	L	Top
L	S	L	S	S	D	S	L	S	S	D	S	
S	L	S	L	S	S	D	S	L	S	S	D	
D	S	L	S	L	S	S	D	S	L	S	S	
S	D	S	L	S	L	S	S	D	S	L	S	
S	S	D	S	L	S	L	S	S	D	S	L	
L	S	S	D	S	L	S	L	S	S	D	S	
S	L	S	S	D	S	L	S	L	S	S	D	
D	S	L	S	S	D	S	L	S	L	S	S	
S	D	S	L	S	S	D	S	L	S	L	S	
S	S	D	S	L	S	S	D	S	L	S	L	
L	S	S	D	S	L	S	S	D	S	L	S	Bottom
G	Ge	A	b	♮	c	ce	d	de	e	f	fe	

Note that S is a semitone, D a diesis, L a limma.

You see that of the twelve different heights of octave there is not one which agrees with another in the arrangement of the elements.

Origins of the fourteen different systems of an octave.

Let us now see how many kinds of systems of an octave emerge from this if the first distinction between octaves, which is the difference in kind,[136] is added in: that is, whether the twelve smallest intervals of all the twelve arrangements can be harmoniously assembled into the seven conventional intervals of a single octave, distributed into eight positions, and in how many ways for each.[137]

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[135] Aristoxenus (*Elementa harmonica*, 36) extended the modes beyond the number of "seven octave scales" accepted by the musicians of his day. Ptolemy (*Harmonica*, Book II, Chapter 11) rejected these new modes.

[136] That is, hard and soft.

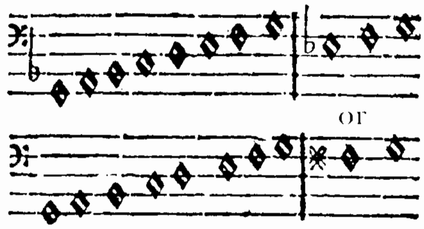
[137] An octave is built out of 3 major tones, 2 minor tones, and 2 semitones. This gives a total of 210 possible permutations. Elimination is effected by Kepler using the rules established in the previous chapter (on the basis of theory and observation) in order to determine the permutations that are acceptable. Kepler is finally left with 14 modes together with 10 variations of these. He regarded his theory of modes as the most general, so that the traditional church modes were included as special cases. On Kepler's theory of modes, see Dickreiter (1978), 170–179.

Here note that *LS.* or *SL.* is the symbol for a major tone, *DS.* or the symbol for a minor tone, *S.* the symbol for a semitone; and diagrams will proceed from low to high, or from the bottom to upper ones.

It is certain, then, that in the octave of G two kinds emerge ac-ding to the two locations of the semitone, in the second or third osition from the bottom.

The first, of the soft kind
LS. S. DS. LS. SD. S. LS.
or S. DS. LS.

The second, of the hard kind
LS. SD. S. LS. SD. S. LS.
or SL. S.



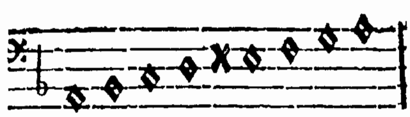
But in the case of *Gϱ* and similarly of *Cϱ*,[138] since two semitones not make up one melodic interval, and are not performed in suc-ssion ἐμμελῶς ("melodically"), therefore the lowest semitone stands one, by laws 9 and 2 of the previous Chapter. Therefore the distri-ution would be:

either like this: *S. SD. SL. S. SD. SL. SL.*, unmelodic by law 3;[139]

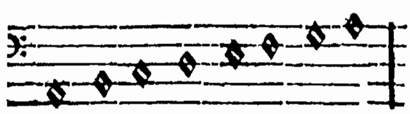
or in the case of *Gϱ* in fact like this: *S. SD. SL. SS. DS.*,

d so on, and this is unmelodic by law 9. Therefore no melodic kind octave begins at *Gϱ*.[140]

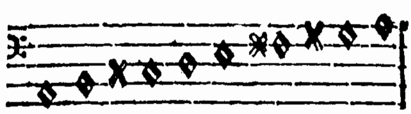
In the octave of *A* in accordance with the three possible locations the lower semitone, the elements will also be assembled into tones three ways: either in this way, *S. DS. LS. S. DS.*, etc., which coincides ith the one which has already been rejected, or in this way, so that om it is born

II. 

The Third
S. DS. LS. SD. S. LS. LS.

V. 

The Fourth
SD. S. LS. SD. S. LS. LS.



The Fifth
SD. SL. S. SD. SL. SL. S.

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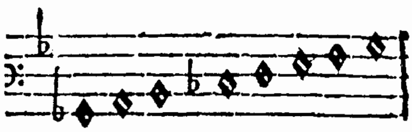
[138] This should be *cϱ*.

[139] Because there are two semitones in the lower fifth, that is, the interval between steps I and V in the scale.

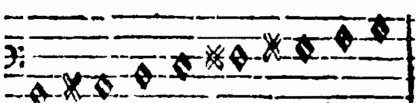
[140] There is, however, a melodious kind of octave with a different distribution intervals, beginning at *cϱ*. See kind VIII below.

V. 

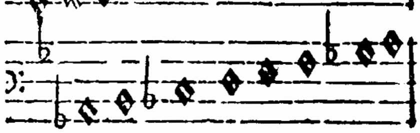
In the octave of *b* only a single assemblage of the elements is possible, which coincides with the fifth form, and the other, that is to say *DS. LS. S. DS. LS. LS. S.* is not allowed because it cannot have a semitone except in the third position from the bottom.

In the octave of ♮ because it is unlawful to conflate two semitones in the second position, by law 9, or to arrange them in succession, by law 2, therefore they must be distributed among the tones[141] in such a way that there is a solitary one in either the first or the second position.

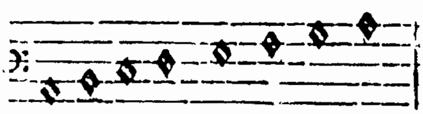
But if it is in the first position, it will follow that the other stands solitary either in the fourth position, which is rejected by the third law, or in the sixth, and four tones will follow in succession in the lower position in the octave, against the fourth law. Hence a semitone must be arranged in the second position, so that there emerges as the sixth kind

VI. 

The Sixth
SL. S. SD. SL. SL. S. SD.

VI. 

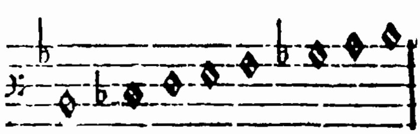
In the octave of *C*[142] according to the two possible locations of the semitone two distributions are possible. The first coincides with VI, *LS. DS. LS. LS. S. DS.*; the other is in number

VII. 

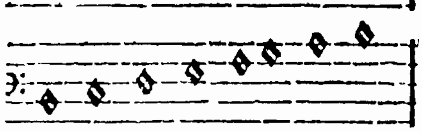
The Seventh
LS. SD. S. LS. LS. SD. S.

Of one form of the octave of *c* we have spoken a little earlier.[143] The other coincides with VIII, which follows: *S. SD. SL. SL. S. SD. SL.*

In the octave of *d* according to the three possible locations of the semitone in the lowest tetrachord, three kinds are possible:

VIII. 

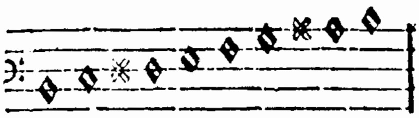
The Eighth
S. DS. LS. LS. S. DS. LS.

IX. 

The Ninth
SD. S. LS. LS. SD. S. LS.

[141] That is, major and minor tones.
[142] This should be *c*.
[143] This form was rejected.

X.

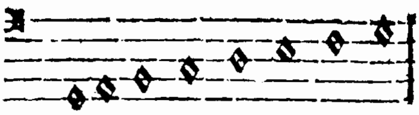


The Tenth
SD. SL. S. LS. SD. SL. S.

In the octave of *dϱ* no melodic distribution can be made; for the semitone would be exiled from the first three positions, which is against the eighth law of the previous chapter.

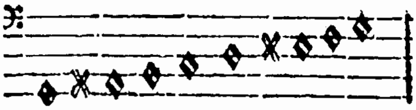
In the octave of *e* again the three possible locations for the semitone in the three lowest positions allow three kinds, among which

XI.



The Eleventh
S. LS. LS. SD. S. LS. SD.

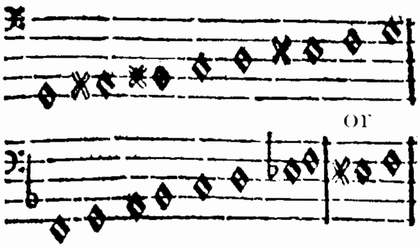
XII.



The Twelfth
SL. S. LS. SD. SL. S. SD.

XIII.

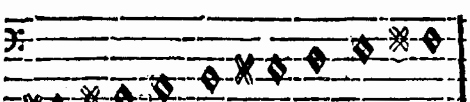
XIII.



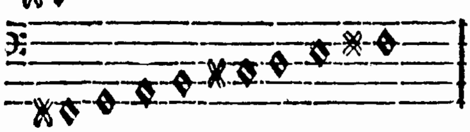
The Thirteenth
SL. SL. S. SD. SL. S. SD.
is the same as the first pattern of the following.

In the octave of *f* the semitone has a single location in the third position; thus one kind emerges, that is to say the same as the thirteenth, in two forms, *LS. S. LS. S. DS. LS. S. DS.* or *SD. S.*

I.



XIV.



Finally in the octave of *f* there are two positions for the semitone, either in the seat next to last, or in the last.[114]

But if, therefore, you arrange them like this: *SL. S. SD. SL. S. SD. SL.*, it will be the first kind, as is apparent by comparison; but if differently, it will be the fourteenth: *S. LS. SD. SL. S. SD. SL.*

Therefore the reason is apparent why in the arrangements[115] II, IV. V, VII, IX, that is in the octaves of *Gϱ*, *b*, *♮*, *cϱ*, or *dϱ*, no unique kind begins.

Second, it is apparent that no genuine transposition can be made, in such a way that the kind of octave remains the same, except the following: I from *fϱ* into *g*, II from *gϱ* into *a*, but in a rejected pat-

[114] This means in the place next to the lowest, or the lowest.

[115] These numbers refer to the matrix of intervals given earlier in the chapter.

tern,[116] III from *a* into *b*, IV from ♮ into *c*, V from *c*ϱ into *d*, VI from *d* into *d*ϱ, in a rejected pattern,[117] VII from *e* into *f*.

But if we bow to the common herd of instrumental musicians who neglect the comma and do not distinguish between the major and minor tones, then all our fourteen kinds of octaves will make up not more than three modes, according to the three positions of the semitone (just as Galilei[118] also says there was a time when not more than three tones were admitted, Dorian, Phrygian, and Lydian)[119]; but they are multiplied into more by pitch alone.

For as far as the third reason is concerned, the structure of the skeleton of the octave, that is still common ground between us and them; and on that account even for us more than fourteen kinds will be born.

Here are the three common arrangements of octaves.

1. With a semitone in the lowest position, which I think was called the Phrygian[150] mode by the ancients, the following of our kinds are equivalent to them.

III. In the octave of *A*.	*S. DS. LS. SD. S. LS. LS.*	Equivalence is genuine with respect to the lowest tetrachord.
VIII. In the octave of *c*ϱ.	*S. SD. SL. SL. S. SD. SL.*	
In the octave of *d*.	*S. DS. LS. LS. S. DS. LS.*	
XI. In the octave of *e*.[151]	*S. LS. LS. SD S. LS. SD.*	Moderate equivalence.
XIV. In the octave of *f*ϱ.	*S. LS. SD. SL. S. SD. SL.*	

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[116] *S, DS, LS, S, SD, LS, LS.*

[117] *DS, LS, LS, S, DS, LS, S.*

[118] Galilei, *Dialogo*, 65.

[119] Ptolemy held this view (Harmonica, Book II, Chapters 6 and 10), though he was skeptical about the idea that they originated with the races whose names they bore.

[150] Kepler here follows the erroneous nomenclature of Henricus Glareanus as a result of which he interchanges the Phrygian and the Dorian modes. The following table shows the definitions of the modes according to Pseudo-Euclid (see Paul [1872], 230–244), the positions of the tones (T) and semitones (S) in the diatonic scale, the corresponding octave in modern notation (cf. note 98) and the names given to the modes by Glareanus.

Ptolemy (*Harmonica*, Book II, Chapter 10) gives equivalent definitions of the modes. For each mode, he defines Mese in the functional system (that is, the note *a* in our notation) by assignment to a note in the positional system (see note 115). This gives for the double octave range in each mode a central octave corresponding to Pseudo-Euclid's definitions, with a fourth above and a fifth below. Ptolemy's system is illustrated in Galilei, *Dialogo*, 64. Cf. Paul (1872), 312–316.

As already remarked, Kepler wrongly assigned his letters to the Greek names of the notes (see note 107). If he had applied his letters to Ptolemy's definitions, however, his modes would have been different from both those of Ptolemy and of Glareanus. For example, the octave of the Phrygian mode would have been *c–c'*, whereas for Glareanus it was *e–e'*, and for Ptolemy *d–d'*. Evidently Kepler had little interest in the correct names of the modes and was content to accept uncritically the names given to them by Glareanus.

II. With a semitone in the lowest position but one, which seems to have been the case in the mode called by the ancients Dorian,[152] the following of our kinds are commonly equivalent.

I. In the octave of *G*.	*LS. S. DS LS SD. S. LS.* *or S. DS. LS.*	Equivalence with respect to the lowest diapente of the system.
In the octave of *f♮*.	*SL. S. SD. SL. S. SD. SL.*	
VI. In the octave of *c*.	*LS. S. DS. LS. LS. S. DS.*	
In the octave of ♮.	*SL. S. SD. SL. SL. S. SD.* *or S. LS. SD.*	
XII. In the octave of *e*.	*SL. S. LS SD. SL. S SD.* *or S. LS. SD.*	Here very little equivalence.
IV. In the octave of *A*.	*SD. S. LS. SD. SL. S. LS.* *or S. LS. LS.*	Moderate equivalence with upper intervals.
IX. In the octave of *d*.[153]	*SD. S. LS. LS. SD. S. LS.* *or S. DS. LS.*	

Footnote 150 (*continued*)

Mode (Euclid)		*Definition*	*Intervals*	*Octave*	*Mode* (Glareanus)
	From	Hypate hypaton		From *H*	
Mixolydian			STTSTTT		Hypophrygian
	to	Paramese		to *h*	
		Paranete hypaton		*c*	Hypolydian
Lydian			TTSTTTS		
		Trite diezeugmenon		*c'*	(Ionian)
		Lichanos hypaton		*d*	
Phrygian			TSTTTST		Dorian
		Paranete diezeugmenon		*d'*	
		Hypate meson		*e*	
Dorian			STTTSTT		Phrygian
		Nete diezeugmenon		*e'*	
		Parhypate meson		*f*	
Hypolydian			TTTSTTS		Lydian
		Trite hyperbolaeon		*f'*	
		Lichanos meson		*g*	
Hypophrygian			TTSTTST		Mixolydian
		Paranete hyperbolaeon		*g'*	
		Mese		*a*	
					Hypodorian
		Nete hyperbolaeon		*a'*	
Hypodorian	or		TSTTSTT		(Aeolian)
		Proslambanomenos		*A*	
		Mese		*a*	

[151] This is the only octave of the set in which all the notes are natural. The others are approximate transpositions employing sharp or flat notes.

[152] In fact, the Phrygian, though the second version is the Hypodorian. The semitone is also in this position in the Aeolian mode, not mentioned by Kepler because he is only concerned with the eight church modes.

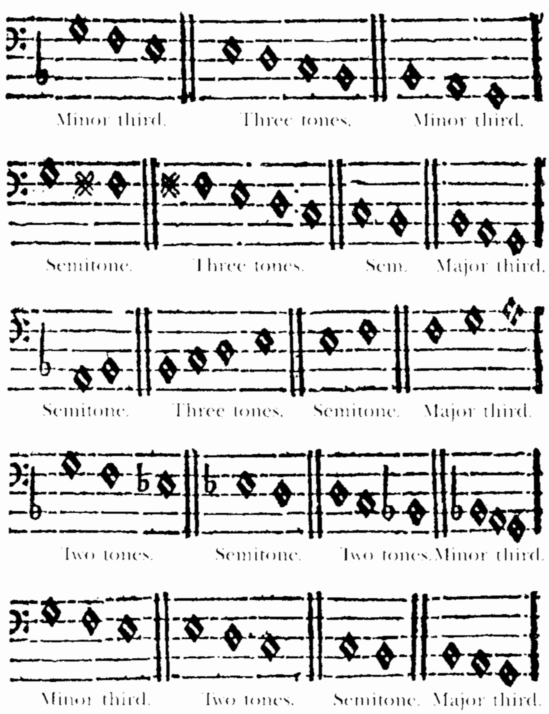
[153] The second octave on *A* and the first on *d* are the only ones in which all the notes are natural. The others are approximate transpositions.

III. With the semitone in the third position, as I believe it was in the Lydian[154] mode, the following of our kinds are commonly equivalent.

II. In the octave of *G.*[155]	*LS. SD. S. LS. SD. S. LS.* *or SL. S.*	Genuine equivalence in the lowest diapente.
VII. In the octave of *c.*[156]	*LS. SD. S. LS. LS. SD. S.* *or S. DS.*	
V. In the octave of *A.*	*SD. SL. S. SD. SL. SL. S.* *or S. LS.*	Less equivalence.
X. In the octave of *d.*	*SD. SL. S. LS. SD. SL. S.* *or S. LS.*	
XIII. In the octave of *e.*	*SL. SL. S. SD. SL. S. SD.*	
In the octave of *f.*	*LS. LS. S. DS. LS. S. DS.* *or SD. S.*	Least equivalence.

Therefore according to the opinion of the common herd on the major and minor tone there can be a great many transpositions[157] of tunes from letter to letter, with the symbols *b* and ✕ as intermediaries.

It remains for us to examine which of our fourteen kinds of octave can become multiple by the structuring of the melodic intervals. For a fivefold distinction between the kinds arises from the contiguity of the tones, as either three tones follow each other in succession, or only two. But if three do, they must either stand in the middle of the octave, unnaturally, having single tones and single semitones on both sides, or incline to one or other end of the system; and in that case there is necessarily a semitone at that end, on account of laws 8 and 3. However if only two tones follow each other, with semitones coming in be-



[154] On the Lydian mode, Kepler is in agreement with Ptolemy. In fact the second version is the Lydian while the first is the Hypophrygian. The semitone is also in this position in the Ionian mode, again not mentioned by Kepler.

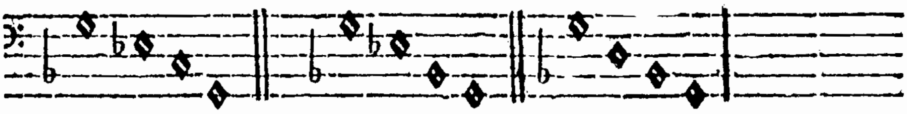
[155] The octave on *G*, with semitone next to the last, is the Hypophrygian mode.

[156] The octave on *c*, with semitone at the end, is the Lydian mode. Apart from

tween, then there is no place for a semitone at the ends; and the two pairs of tones stand either at the top or at the bottom, with a soft third at the opposite end, if at the bottom, the right way up, having a semitone above, or if at the top, the wrong way up, having a semitone below.

On this basis among our fourteen kinds are found interspersed another ten, coinciding in the first four positions with the original ones, or even in five. However where they begin to differ, the conjunction "or" has been put in front.

But if the final reason which differentiates the tones is now added, that is to say τονή and πεττεία, the number will be tripled, and there will be 72 kinds. For every skeleton of an octave out of the whole 24 has both a third and a fourth [note]



consonant with its first, and a fifth and sixth. Then if these elements of melody are employed chiefly round the fourth and sixth, then they set up a diatessaron in the lowest position; but if round the third and sixth, there is a diatessaron in the middle; but if finally round the third and fifth, the diatessaron is in the highest position.

I do not however say this because such hair-splitting is necessary, for I know that very often three patterns are mixed in one melody; but so that from this number a judgement can be passed on the distinctions between tones which the moderns proclaim. The distinctions are partly such that they seem to me to be able to establish as many modes as there are tunes altogether; and the rest fit into a single tone. But if we observe these basic principles, the number of tones, however large, is nevertheless finite. To sum up, the tones as differentiated by real boundaries and distinctions, not in respect of the height of the note on the instrument, but also in the actual human voice which can make the basis of a natural system at any note, high or deep, are either only three, or fourteen, or twenty-four, or seventy-two.

With these my basic principles, I reconcile the eight common tones, which they call ecclesiastical; and according to the ancients, in the opinion of certain people[158] they were:[159]

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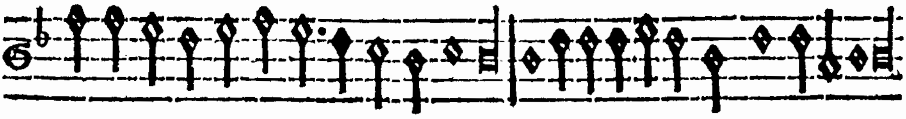
the octaves on *G* and *c*, the octaves in the set are approximate transpositions employing sharp or flat notes.

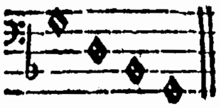
[157] When the distinction between major and minor tones is removed, the transpositions become exact.

[158] Following Glareanus, whose modes are illustrated in Galilei, *Dialogo*, 78. See also note 150. In the following notes, when references are made in connection with Glareanus, it is to be understood that the names are those given to the modes by Glareanus.

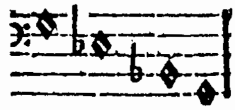
[159] The examples of the church modes given here were often used in textbooks. See, for example, L. Lossius, *Erotemata musicae practicae* (Nuremberg, 1570), Book I, Chapter 7. The third example, from Orlande de Lassus's motet, "In me transierunt," has already been quoted by Kepler.

I. The Dorian[160] II. The Hypodorian

Adam the first man. *Noah the second.*

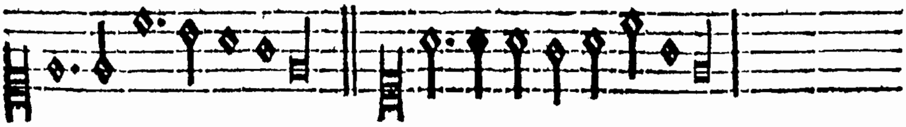
These are the two kinds of soft melody, from the kind of octave which I have put as first among the fourteen. I see no other difference but purely in the pitch, which in the first is higher, in the second lower,[161] unless you wish to distinguish them through the final reason,



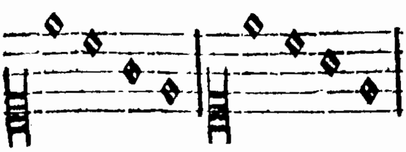
on the grounds that the first observes this skeleton, the second this.



The third the common Phrygian[162] The fourth the common Hypophrygian.



Abraham the third. *The four Evangelists.*

Nothing is more obvious than that these two of the hard kind[163] belong to our kind XI, and should be written from *e*, and again there is scarcely any distinction other than of *pitch,[164] unless you wish to fit these skeletons to them separately.

* The pitch of a tone is threefold. 1. Absolute and in a sense material, as in the human voice and also on instruments if the point of origin of the system ascends from G to g, in which case the pattern of the system does not change. On this nothing at present. 2. Formal, of the actual point of origin and of each tone in the natural system formed by combination of two diapasons. This is not meant at this point either; for it introduces the modulation of tone when the tonic key of the system has been changed. 3. Again formal, the pitch of the melody with reference to its own tonic key. While that remains the tone also remains the same, but takes on, so to speak, another shape, so that it does not go up to the octave of the tonic, but keeps below the sixth, fifth or fourth [note], or frequently dips below the tonic, though returning in the end to the tonic. Thus these skeletons should not be understood to mean that the melody runs through all the links or notes illustrated here. It is sufficient if it runs through some, or those which are an octave below them. Therefore we could with the ancients call those which dip below the tonic in this way plagal, the rest authentic.[165]

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[160] In fact, the Phrygian.

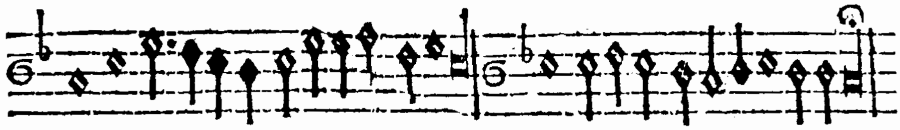
[161] According to Glareanus, the Dorian mode starts on d and the Hypodorian on *A*, both modes proceeding through ♮. Kepler begins his system I on *G* but proceeds through *b* and in the second case also through *e* flat. A comparison of the sequence of intervals shows that the first case of Kepler's system I corresponds to the mode named Dorian by Glareanus, and the second case to the mode named Hypodorian.

[162] In fact, the Dorian.

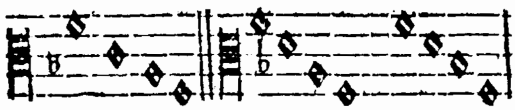
[163] According to Kepler's principles, these should be soft, as they have minor thirds and sixths.

[164] Glareanus begins the Phrygian mode on *e* and the Hypophrygian mode on *H*, so that the position of the semitone differs in the upper tetrachord.

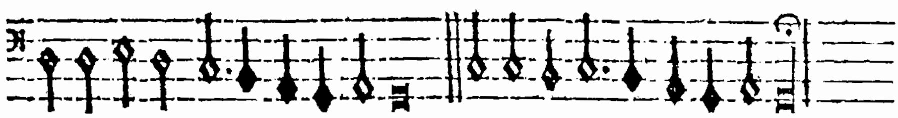
[165] There were eight traditional church modes divided into two classes called au-

The fifth, the common Lydian.[166] The sixth the common Hypolydian.[167]

The five books of Moses. *The six appointed urns.*

And these similarly are obviously to be written starting from *f*.[168] They are of the soft kind,[169] and are differentiated by pitch or by these skeletons from our XIII.

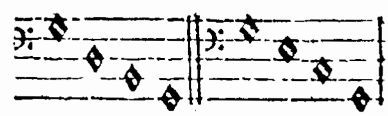


The seventh, the common Mixolydian. The eighth, the common Hypomixolydian.



The seven liberal arts. *And there are eight parts.*

Although the plan of the inventors is obviously that they wanted to write the first and second starting from *d*, the third and fourth from *e*, the fifth and sixth from *f* and the seventh and eighth from *G*,[170] yet it is certain from



[auth]entic and plagal. The authentic modes were the octaves starting on *d*, *e*, *f*, and *g*. [In] these modes respectively, the notes *a*, *c*, *c*, and *d* were used as reciting notes in plainsong, the cadence falling on the first or key note. When these octaves were arranged to start on *a*, ♮, *c*, and *d* respectively, so that the key notes came in the middle, the corresponding modes were called plagal or oblique. In order to avoid having the reciting note at the extremes, however, a new reciting note was chosen, at an interval of a third above the key note, except in the case of the plagal mode beginning in *d*, when *c* was chosen as the reciting note instead of the unsatisfactory ♮. Glareanus added four extra modes, consisting of two authentic modes and the corresponding plagal modes. To the new authentic modes, beginning on *a* and *c* respectively, he gave the names Aeolian and Ionian. These resemble the old plagal modes which he called Hypodorian and Hypolydian, and have survived as the modern minor and major keys respectively.

[166] In fact, the Hypolydian.

[167] In fact, the Lydian.

[168] Glareanus begins the Lydian mode on *f* and the Hypolydian on *c*. Although Kepler also begins the Lydian mode on *f*, his intervals do not agree with those of Glareanus, for Kepler proceeds through *b* while Glareanus proceeds through ♮. The intervals of the second alternative of Kepler's type XIII, however, agree with Pseudo-Euclid's definition of the Lydian mode. The first alternative gives intervals in agreement with Glareanus' definition of the Hypolydian mode.

[169] According to Kepler's principles, these should be hard, since they have major thirds and sixths.

[170] Glareanus begins the Mixolydian mode on *G* and the Hypomixolydian mode

the arguments which we have presented above that these also in the hard kind along with the first and second should be written starting from the note *G*, with which these skeletons from our kind II agree.

Therefore by this application of the common eight tones to my kinds I, XI, XIII and II of octave, I can expose much more clearly the distinction between these tones than our musicians, because they despise the comma. For those who write them starting from *G* have all four of the third, fourth, fifth, and sixth in legitimate consonance with the first; whereas those who start from *F* and *e* are using impure consonances, in the lower part a wide fourth *e–a* and *f–bb*, and a wide fifth *e–*𝔥𝔥 and *f–cc*.[171]

What, therefore, becomes of the remaining ten kinds, you will say, since all the conventional tones are reduced to four? Of course the purpose they serve is for musicians to know, that if we temper as nature tells us the systems of the basic octave, depicted as starting from *G*, then it will in no way be possible so easily and in so many different ways to transpose a melody without altering its purity, as they themselves do in practice. They are therefore free either to reject all the remaining ten, and with them their transpositions, or to confess that more patterns of tunes, indeed more modes or tones can exist, and that they are different from those adopted, some more and some less, in accordance with the equivalences which you see annotated in the margin above. For if we take no account of the upper intervals of a single octave, certainly a number of transpositions are allowed from *G* to *c*, from *A* to *d*, and others.

Finally you may ask what prevents these transpositions which I have already rejected so often? For the ears seem not to prevent them, since it is confessed that a difference of a comma between intervals which are in separate positions in the system cannot be distinguished by the ears. For let there be a system as well tuned as possible, according to the laws of nature so far explained, let *G–a* and *a–*𝔥 be struck, and let all men who can hear be asked which of these intervals is the greater. They will say they do not know; and they will not tell the difference before the intervals have been put together in the same position in the octave, that is, when a distance has been marked out on the string with compasses to delimit one eighty first part of it, and thus two lengths of 80 and 81 have been struck in successive instants and compared with each other.

I reply that although the hearing does not distinguish when only three strings of the octave are struck (the terms of the two intervals), yet when all the strings in one octave are struck, in that way it does

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on *d*. Kepler's intervals only agree in the case of the Mixolydian. Ptolemy's Mixolydian mode begins on *H*. Though not accepted by Ptolemy, the Hypomixolydian mode was one of the 13 modes recognized by Aristoxenus. Aristoxenus' modes are illustrated in Galilei, *Dialogo*, 57.

[171] The fifth *f–cc* is in fact perfect (see note 122).

eventually distinguish the first born and natural octave of *G* from the octaves of the other keys. For there are in all the kinds of octave the same seven melodic intervals; and when they have once been evinced by the striking of the strings, they at once stick closely in the memory, so that it is easily clear to the hearing in which position in the octave each natural tone begins. On this basis appreciation of a comma will be implicit in the act of distinguishing between the octaves. For just as the hearing approves the consonances themselves and all the melodic intervals by its effect, though it does not reckon the lengths of the strings, which provide the cause of the consonances, so the same hearing also notices the effect of the comma, 80:81, even though it does not reckon it, and therefore does not detect it as a separate sensation in practice.

Chapter XV.

Which Modes or Tones Serve Which Emotions.

So far I have spoken of the basic principles according to which the tones come about and are distinguished from each other. Now I shall say a little about their effect, though it will indeed be in agreement with my principles.

The reader must first be warned that I am not dealing with the difference in the states of mind which make a man sing rather than use plain pedestrian speech, nor those which follow in the hearer, in the one case from song, in the other from simple speech. For just as there is an eagerness of mind which precedes every song in every way in the singer, and so to speak dictates the air to him, in the same way every kind of melodic tunes is followed by pleasure in the hearers.

We must separate this general state of mind, or rather, as a topic it must be made subordinate to that variety of emotions which we are enquiring into here.

Now since all components of melodic and natural melody are employed to stir up emotions, in imitation of the sounds which animals give out to testify their desires, from that it is clear that this enquiry is various and manifold, and very nearly infinite. Since it is too much for my muscles, it would be more correctly passed on completely to the practical men, that is, to practicing musicians, seeing that without teaching, guided solely by nature, they emerge time and again as the authors of wonderful tunes. That is far easier for them than making long and fluent speeches on what melody is, and in what it consists, or how it has come about.

For as ordinary speech is represented in poetry, so the voices and gestures of animate beings are represented and, so to speak, depicted in melody; and hence certainly as the poetic faculty, so also the musical faculty of composing melody is to be learnt by practice and exercises alone, if talent for it has also been provided by nature.

However, it is part of the most honorable recreations of the mind to track down the causes of things, or to examine at close quarters those which others have found, so much indeed that even Aristotle devoted by no means the smallest part of his *Problemata*[172] to theoretical study of harmony. Then here also something must be attempted, and the boundaries of philosophy must be advanced, and the infinity of that enquiry limited by applying the laws of method, and brought

[172] Aristotle, *Problemata*, Book 19, 917 b 18–923 a 4. See note 107.

under general headings, so that through particular instances the same judgement can quickly be applied to similar cases.

I. The elements of melody itself, and of the emotions which follow the kinds of melody, are proportional, and almost the same in number in each case. For even the actual names of the components of melody, ἀγωγή, τονή, πεττεία, πλοκή, allude to certain emotions. The first serves for simplicity, the last for luxury. The former is like the body, the latter like colors. The uniformity of *Tone* takes the attention; *Pettia*, or gaming, is employed for delight and recreation. These points are general for all musical modes.

II. On *Agoge* Galilei makes this special point,[173] that there are two different kinds of it, the one upwards, the other downwards. The former of these serves for joy, the latter for sorrow and weeping. The cause is natural: for below a low note is given out with a slow motion, above a high note with agitated motion. Therefore when the note descends, it is coming towards rest; when it ascends, its motion advances, because in choral melody we very often leave off on the lowest position. In the former case therefore the note is weakening, in the latter it is strong. But in sadness also the mind is weak, and so are all its functions: in cheerfulness it is lively and active.

III. The force of a leap is also great, as it is like a potential *Agoge*; for it has rashness, movement, boldness, it is warlike, manly, brash, if it is frequent, especially over a diapente. Its figure, the triangle, consists of acute angles, and covers the whole circle in three lines. On the contrary, a single ascending leap over a soft sixth, with a downward *Agoge* following, expresses the magnitude of grief, and is suitable for wailing, on account of the similarity of the note, as in Orlando's "In Me transierunt." [They have come over to Me.][174]

IV. It also makes a great difference what height in the system of the octave, from the lowest point of origin, the greatest part of the melody occupies. For if it ranges over the whole octave, or beyond, the melody is spirited; but if only a diatessaron, mild and agreeable. A diapente has moderation; a soft third dejection and faintheartedness. Some people attribute this to a descent below the point of origin over a diatessaron; and they designate tones of this pattern "oblique,"[175] or subsidiaries of their authentic tones.

V. Among the general elements must be included also most particularly the quickness or slowness of the rhythm, or beat. Of these quick-

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[173] Galilei, *Dialogo*, 76.

[174] Kepler has already quoted Orlando's motet, with a musical illustration, in chapter 13.

[175] That is, plagal. The relationship between the authentic and plagal tones is described in note 165.

ness is suitable for anger, movement, battles, joyfulness; slowness for feelings which we enjoy peacefully, sorrow, love, desire, pleasure of possession.

VI. Add also the mode, that is the actual rhythm: for the force of the one called triple is different from that of the one called double. Much of its force happens to be drawn from imitation of leaping dances: the alternating variety of motions in them, since it is plain to all, at once informs the listeners, as if bringing them into what is going on. Then the triple rhythm is turbulent and lively, the double quieter, more peaceful and milder. But the rhythms and their symbols have nothing to do with this book; so let us say goodbye to them.

VII. We must now come to the features by which we have said the tones are properly differentiated; and the most important of them is the variety of kinds of melody. And here a most plentiful crop of controversies springs up, over which of the ancients' three kinds[176] is attributed to which emotion. This we shall pass over untouched, since neither is the distribution of melody into kinds which are three in number natural, nor are the ancients consistent in this matter. My subject will be the natural two kinds of melody, that is to say hard and soft; and the names themselves proclaim what emotions they stir up. For as woman is made chiefly to be passive, man to be active, especially in the act of generation, so the soft kind is fitted for the feminine emotions of the mind, the hard for masculine activities: the differences between these will become clearer from what follows.

On the Enharmonic Aristotle writes that by it minds are filled with divine passion; on the other hand the Spartans' magistrates because this was held to be an effeminate and counterfeit kind of melody, branded the man responsible for its acceptance with ignominy and drove him from the city.[177]

VIII. For not only the kinds of melody but also the tones in every case differed in the location of the semitone. This location of the semitone gives spirit and feeling to both the kind and the mode or tone; however one argument must be applied to this location in respect of the kinds, another in respect of all the tones. For we have said that the kinds arise first of all in the principal octave, which starts at *G*, in which the location of the semitone in the lowest but one position forms the soft kind, in the position next before the last but one, the hard kind. In the former case there is a soft third in the lowest position, in the latter a hard. What, then, is the connection between this location and the feelings? Or what is there in common between the minor third and women, and passivity, and softness; and what between the hard third and masculinity, hardness, activities, effectiveness?

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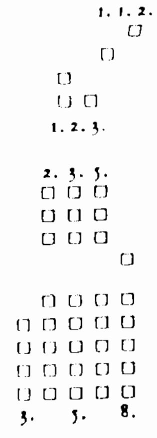
[176] These are the diatonic, chromatic, and enharmonic genera.

[177] According to Aristotle (*Politics* 1340 b 3 and 1342 b 3–20), it is the Phrygian mode that inspires divine passion. The story of the banishment of Timotheus of Miletus by the Spartans is also related by Boethius (*De institutione harmonica*, Book I, Chapter 4), though his offence was not the introduction of the enharmonic genus but the support of the tempered chromatic species. For a description of these scales, see Boethius, ibid., Book V, Chapter 16. Kepler's source may have been Zarlino, whose inaccurate account is corrected by Galilei, *Dialogo*, 102–103. Plato alludes to the emotions evoked by the Greek modes in the *Republic*, 398E–399A.

First, then, you will remember that the hard third arose from the pentagon and the pentagon uses the division in extreme and mean ratio, which forms the divine proportion. However, the splendid idea of generation is in this proportion. For just as a father begets a son, and his son another, each like himself, so also in that division, when the larger part is added to the whole, the proportion is continued: the combined sum takes the place of the whole, and what was previously the whole takes the place of the larger part. Although this ratio cannot be expressed in numbers, yet some series of numbers may be found which continually approaches nearer to the truth; and in that series the difference of the numbers from the genuine terms (which are not countable but inexpressible) by a wonderful coincidence breeds males and females, distinguishable by the members[178] which indicate sex. Thus if the larger part is in the first place 2, and the smaller 1, the whole is 3. Here plainly 1 is not to 2 as 2 is to 3; for the difference is unity, by which the rectangle of the extremes 1 and 3 is less than equal to the square of the mean, 2. Then by adding 2 to 3 the new total becomes 5; and by adding 3 to 5 the total becomes 8, etc. The rectangle of 1 and 3 creates a female, for it falls short of the square of 2 by unity; the rectangle of 2 and 5 a male, for it exceeds the square of 3 by unity; the rectangle of 3 and 8 a female, for it falls short of the square of 5 by unity. Again from 5 and 13 arises a male, in respect of the square of 8; from 8 and 21 a female, in respect of the square of 13; and so on infinitely.[179]



That is the nature of this division, which relates to the construction of the pentagon, and God the Creator has shaped the laws of generation in accordance with it, in fact in accordance with the ratio of inexpressible terms which is genuine and perfect in itself, the logic of the seeding of plants which have been commanded individually to have their own seed within themselves, but in accordance with the combined ratios of pairs of numbers (of which the falling short of one by unity is compensated by the excess of the other) the coming together of male and female: what is surprising then if the progeny of the pentagon, the hard third of 4:5 and the soft 5:6, moves minds, which are the images of God, to emotions which are comparable with the business of generation? Here must be repeated from Chapter III that although 1:6 comes from the hexagon, yet its remainder 5:6 is consonant not on account of the hexagon but on account of its derivation from three tenths of the circle, by the doubling and halving of terms. Thus this remainder also, and its progeny the minor third, comes from the pentagonal class of figures. On establishing, then, that the

[178] That is, the terms.

[179] This is the Fibonacci series. As the series progresses, the ratio of adjacent terms approaches that of the Golden Section associated with the pentagon. Kepler had explained the symbolism in a letter to Joachim Tanckius on 12 May 1608 (KGW 16, p. 157).

association of the two thirds represents the association of male and female, it is now no trouble to assign to each sex its own third. For the major third will turn out manly, the minor feminine, since the proportion of their actual bodies is also the same as that of the powers of both body and mind. And since the major comes from a figure with an uneven number of sides, that is to say the pentagon, but the minor originally from the decagon which has an even number of sides, it is also in agreement with the views of Pythagoras,[180] who said that uneven numbers were male, and even ones female (which is confirmed by this study of excesses and shortfalls, since the uneven is also in excess), so that the former is considered of the masculine sex, the latter of the feminine.

In addition to these reasons there is the consideration of the melodic intervals, the smallest of which is the semitone; for a semitone following after always invites the voice to climb over it, on account of its small size; for it is like a crest on a slope which gets more gentle. And every time a semitone occurs towards the upper part, it is taken as a sort of boundary to the melody, towards which it tends, and then as if the crest has been passed, and when the effort is completed it often begins to turn back to the lower part. Certainly if we sing *RE, MI*, the hearing is not satisfied, but expects that *FA* should also be added. Therefore since a hard third, which has the lowest position in the eighth tone, lacks a semitone, which is only added to make up a diatessaron, it is deservedly considered as active, and full of efforts, having force which is γόνιμος [productive], and ἀκμὴ ἄοχετος [irrepressible vigor], seeking its own end, that is to say a diatessaron, of which the semitone is like an ἔκχυσις [bursting out] for it, sought with its whole effort. But the minor third which stands in the lowest position of the first tone, since it has encompassed the semitone, from which it usually falls back when it has climbed over it, as if content with itself, and made by nature to be overcome and to be passive, always like a hen prostrates itself on the ground, ready for the cock to tread it. And there are the causes of the emotions in the kinds, and in the tones of the primary system, which rises up from *G*.

Now let us consider the same location of the semitone also in respect of all the tones indiscriminately. Here I shall adapt my nomenclature, as I have also done up to now, to the common tones, as I have picked them out in accordance with my basic principles. For as to the rest of the ten kinds of octave, which I added for the sake of comparison, it shall be for the musicians to judge whether they deserve to be called tones.

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[180] The theories of the Pythagoreans are reported by Aristotle. See, for example, *Metaphysics*, 985 b 23–986 b 9. Cf. Heath (1949). However, the association by the Pythagoreans of odd and even numbers with male and female respectively is reported not by Aristotle but by Alexander in his commentary on 985 b 26. Alexander Aphrodisiensis, In *Aristotelis Metaphysica commentaria*, ed. M. Hayduck (Berlin, 1891), 39.

It has often been said, naturally indeed, that of the melodic intervals the larger demand that the lowest position be given to them, so that large may be associated with large. For low notes also, which are similarly at the bottom, are determined by some magnitude, that is, by long strings, and high notes by short ones; or, which is the same thing, a low note is given out by a large, slow motion, a high note by a small, quick one. The sequence is therefore natural when in the perfect tetrachord there is a major tone in the first position, a minor tone in the second, and a semitone in the third and highest. But when everything goes in accordance with nature, we are glad: therefore the tones which have their lower tetrachord split up in that way are glad. Now the seventh and eighth have them in that way, which people think were designated by the ancients, with remarkable inconsistency sometimes Phrygian and sometimes Mixolydian, though I am more inclined to believe that they were rather called Lydian by the ancients. For they testify of their Lydian[181] mode that it fills minds with a divine passion, that is with eagerness and warlike spirits, just as do our seventh and eighth.

When, therefore, things are upside down, so that there is a semitone in the lowest position, which happens in the ecclesiastical third and fourth, which were Phrygian to the ancients, it agrees with this reversal of the order of nature that they sound plaintive, broken, and in a sense lamentable. But when the semitone is in the middle position, there is a middle emotion of tranquillity, gentleness, and enjoyment of talk and storytelling. For these the first and second are fitted, that is to say those of the soft and feminine kind, which are believed to have been formerly called Dorian. Vincenzo Galilei certainly describes it thus as Dorian,[182] because it is by nature steadfast, calm, without violence, fitted for dignity and seriousness, which is more or less true of these two. For the tones which are called even-numbered should always be softer, inasmuch as they are subsidiaries[183] of their authentic tones, which are named from their uneven number, for the reason which was adduced a little above in the case of Number IV. Thus Galilei also makes the oblique tone of the first languid, tearful, and timid; for movement towards lower notes belongs to languor, towards higher notes to vigor, though I myself should prefer both to temper those epithets and to distinguish the reason. For languor, tears, and fear are not the property of the Dorian tones but in general of sallies towards low notes and frequent descents below the point of origin.

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[181] Earlier Kepler has erroneously called the Lydian mode soft. The description of the Lydian mode given here seems to suggest that his earlier classification of the third and fourth as hard and the fifth and sixth as soft was a simple mistake and unintended.

[182] Galilei, *Dialogo*, 62.

[183] That is, plagal modes.

IX. So far the location of the semitone has been discussed; but we have not yet assigned properties to all the tones. The next thing therefore is for us to speak also of the imperfect and impure consonances, with which the skeletons of the octaves are structured. And first indeed their essence, so to speak, will have to be examined, and then their location in the system.

Since, then, the natural and original disposition of the system of the octave is that in which the third, fourth, fifth, and sixth are in perfect consonance with the lowest string, such a tone arouses in the mind everything which is in accordance with its nature, the soft kind in fact passivity, the hard activity, or emotions which fit them. This property agrees with the first and the eighth which arise from *G*; and on that account Galilei[184] was not wrong to assert that the eighth coincided with the first. Be it understood, however, that each preserves the character of its own kind or sex. Thus the seventh and the eighth stand at the head of the others on two accounts, both because of the location of the semitone, and also on account of their perfection.

On the other hand, the fifth and sixth,[185] and the third and fourth[186] employ consonances in their lower parts which are impure and augmented, the former the diatessaron and the latter the diapente. The fact relates to them the force of sorrow and emotions which depart from temperate gentleness.

Now in the fifth and sixth, the diatessaron is in fact ὄρθιος [shrill], with the natural position of the semitone in the highest position, no less than in the seventh and eighth; but there are two major tones in the lowest position, and after them a semitone, the conflation of which forms a diatessaron which is over size by a comma.

Thus those modes promote grandness of emotion, such as devotion, wonder, worship, sorrow; or again hope, confidence, a kind of elevation of the mind above its immediate fortune.

However in the third and fourth, to the inverted form of the actual diatessaron[187] is also added this combination of consonances, which increases the sadness and languor of the mind.

But if we may follow the analogy beyond the conventional tones, to the third of my kinds, of soft melody, which is written from *A* because it has in fact a semitone at the bottom, but has a perfect diatessaron, therefore I should connect it with pleasurable sadness, as

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[184] Galilei, *Dialogo*, 63.

[185] Kepler begins these on *f*. The interval *f–a*, as Kepler noted earlier but does not mention here, is a wide major third, while the diatessaron *f–bb* is also wide.

[186] Kepler begins these on *e*. The diapente *e–*𝔥 is pure, as also is *f–cc*, though Kepler thought it was wide. However, the diatessarons *e–a and f–b*, which he does not mention here, are also wide. Kepler's reference to the diapente is puzzling, as the only imperfect diapentes are *a–e* and *b–f*, both of which are narrow but not relevant to the present issue.

[187] In these cases, the semitone is in the lowest position of the diatessaron.

when we are pleased with softness of mind, with loves and desires, or when joy expresses itself in tears.

So much for consonances in their own right: now let us see what effects they have in relation to their place in the system.

Where, then, τονὴ and πεττεία are mostly employed round the fourth and sixth strings, in such a way that a diatessaron appears to have been established by them in the lowest position, because that is the interval of the tetrachord, which has encompassed all three kinds of melodic intervals, and is therefore particularly concerned in this characterization of the tones, in that case this pattern invites the melody to submissiveness, for the same reason as I have adopted above for Number IV. But if these components of melody are employed round the third and fifth, establishing a diatessaron higher up, the melody is also excited and elevated, especially if it sallies forth to the octave, as the νόμος ὄρθιος ["orthian strain"][188] of the ancients seems to have been. Lastly, if it is around the third and sixth, so that the diatessaron is brought to the middle, which happens more often in the first and second than in the seventh and eighth, a middle emotion will also be evoked. However, these structures of skeletons are often mixed with each other, so that they are different in different parts of the melody, according to the variety of the text and the ingenuity of the craftsman who is painting the text in melody.

X. But if it is true that certain people make tones of uneven number, I, III, V and VII higher, and those of even number, II, IV, VI, VIII,[189] lower, also in respect of the point of origin of the octave, certainly height will correspond to excitement of the mind, depth to dejection. In that case we shall certainly be able to distribute the emotions of the four types into eight types, according to the same number of tones, so that the seventh has anger, violence and strength; the eighth has cheerfulness, eagerness, and sharp pleasure; the third has sharp pain and desire; the fourth has weeping and love; the first has merriment, weddings, and feasting; the second moderate cheerfulness, conversation, and story-telling; the fifth panegyrics, acclamations, confidence, hope; the sixth devotion, great sorrow, and so on.

And now it is time for the disquisition to return to its basic starting point. For one thing of which the philosophical reader must be warned is that our musicians express all emotions promiscuously in all tones. They do that because all melody promiscuously gives birth to pleasure in the listener, and demands eagerness in the creator; and they are able to do that because there are several devices for stirring up the emotions, partly listed at the start, which can be applied anywhere through all the tones promiscuously. However, if they take pains

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[188] See note 127.

[189] Glareanus, for example. See Galilei, *Dialogo*, 78.

to apply every safeguard to help in producing the same effect, they will not readily neglect the delights of the tones, which are appropriate for suggesting emotions. However, let them see to what is permitted to them in practice; for me it is enough to have progressed to this point in theory.

Chapter XVI.

What Harmonized or Figured Melody Is.

Although the word "harmony" is used by the ancients for "melody," yet we should not understand under that title composition in several parts which are in harmonic consonance. For it does not require much proof that the performing of several parts together in constant variation of harmonies was an innovation, unknown to the ancients. On this point consult Vincenzo Galilei in his Italian work on music.[190] It is indeed often quoted as an objection that harmonized melody was banished from the Platonic Republic[191] as if it had been in use then; but the passage is to be understood as referring to instruments, to the pipes, bagpipes or lute, when one note is sounded either constantly, or intermittently, giving way when a dissonant note approaches. In that method the ancients showed no greater skill than our bagpipers.

In these last centuries this method of performing has begun to be called "figured,"[192] because the first authors of it did not make the notation as simple as in choral music, but used various shapes, colors, and points. Of these symbols there are those which direct silence,[193] and those which direct a note, some a long one, some a short one; there are those which are used for distinguishing between tones, and those which are used for distinguishing between rhythmic measures, for fugues, for repetitions, and the like.

We therefore define the performing of parts together in harmony as follows. It is when there are two, three, four, or more parts, or melodic and coherent tunes, as we have described in Chapter XIII, all of the same kind, and of the same or related modes, running at the same time, so that they make consonances which are either pure or leavened with a very brief interpolation of melodic dissonances; yet they are not constantly following an identical course, nor the same in succession, but differing in an actual alteration of successions so as to give delight. For just as the consonance of two voices is to unison in proportion, so harmonized melody is to the simple tune of one voice. On this principle all the limbs of the definition are based, as I shall proceed to explain.

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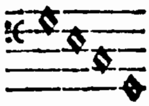
[190] See Galilei, *Dialogo*, 104–105.

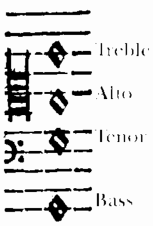
[191] Plato, *Republic*, 399C-E.

[192] Contrapuntal music in which the parts move independently, with short notes in one part, for example, against long notes in another. This may be contrasted with note-against-note counterpoint.

[193] That is, rests.

I. First a point which must be remarked on about the plurality of voices is that the practitioners have established it in a way agreeing as far as possible with nature that although a great many parts often fit together yet all of them are designated by only four names, in accordance with the number of perfect harmonic means in one perfect system of a diapason. For they name the highest part the Treble, the one next to it Alto, the upper of the low parts Tenor, and the lowest Bass. For between the two notes which sound a diapason there are only two means which are in harmonic consonance at the same time with the extremes which are in identical consonance, as explained in Chapter III.[194]



Although the origin of those terms is Italian, it seems to refer rather to a composite system, so that in it the Alto and Bass keep to opposite ends, though the former is not indeed the highest, but because these two frequently (though not at the close) make a diapason, the former indeed being upper and higher, the latter deeper and lower. Thus the origin of the term gives to the Tenor the idea and simple rhythm of the melody; the Treble seems to have been given the name "Discant" from its constant changeability and straying from its area, which is at the opposite end of a diapason from the Tenor. Thus if we compare the components of melody which were adduced in Chapter XIII from Euclid, even though they all belong to all the voices, yet the Tenor is mostly employed in ἀγωγὴ the Treble in πλοκὴ, the Alto in τονὴ, the Bass in leaping through harmonic intervals. However πεττεία is common to all four, though more prominent in the Alto.

These four properties of the voices have been drawn from the actual nature of things. For as by the declared definition singing in parts must constantly change from one note to another, it was necessary for that to be done in such a way that one of the intermediate parts, in comparison with the others, should be occupied in a narrow space within the system. For if they all had equally slipshod liberty, it would necessarily follow that the lower part would frequently stray into the territory of the upper, and the other way round; and there will be insane confusion, and the distinction between the four voices will be rendered pointless. However these narrow territories should preferably be held by one of the middle ones (Alto or Tenor) rather than by the extreme ones, Treble or Bass.

For if that sort of thing had happened in the extremes, every variation in the intervals would have been towards only one direction, say

[194] For the harmony of all the planets, Kepler needs six parts. Yet the system of four parts, to which he here refers, was regarded as the ideal by the musicians of his time and had been described by Glareanus as the perfect art.

either to the high or to the low. It was therefore better for the part singing to be fixed or tied in a narrow situation in relation to one of the intermediate voices, as if one of the parts were its own boundary, but for the variation in the intervals to tend sometimes upwards from it, sometimes downwards. However, the lower one of the middle voices ought not to have been restricted, because the Idea of the melody, which is ascribed to the Tenor, the lower, of the middle voices, should also be free, and pass through the middle section of the whole system, so that the surrounding voices are nothing but its colors and ornaments or decorations.[195] However, the lower intervals are always larger than the higher ones of the same name.[196] Then for the Tenor to have a position intermediate in size it had to be the lower of the middle voices, and thus the Alto[197] had to be reduced to a narrow space not the Tenor. Therefore wandering abroad is left for the Treble and Bass, but with this difference, that the Bass, which sounds a low note, is expressed with a large movement and long strings, and with slow beats as well, also roams with large and therefore harmonic intervals; the Treble, which sounds a high note, flutters overhead, and is produced by short strings and by a quick movement in a narrow space, and also with short and therefore more frequent beats, and roams everywhere with the smallest melodic intervals. These, then, are the properties of the voices. I shall now also clarify the remaining articles of the definition.

Now as far as that primary and chief seasoning of tuneful melody, dissonances, are concerned, first they must be drawn not from any ἀναρμόστος ("unharmonious") interval whatever, but from among the melodic intervals. For because individual parts performed together in harmony admit only melodic intervals between notes which follow each other in succession, and are moreover of the same kind and tone, as was stipulated in the definition, it is readily evident that if any single harmony is set up between such parts (since all harmony is analyzed into melodic elements), it cannot ever come about by any wandering of individual parts from the point of origin of their octave that two of them make any interval which is other than melodic, though it may not always be consonant. Second, although leading practitioners oc- Dissonances.

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[195] Kepler is evidently thinking here of the old idea of the Cantus firmus, as it had been described by Martin Luther. In this form of singing, a plain melody (the cantus firmus), sung by the tenor, was wonderfully embellished and decorated by the addition of the other parts. See Dickreiter (1973), 185.

[196] It is not clear exactly what Kepler meant by the statement that the lower intervals were larger than the corresponding higher intervals. The most likely interpretation seems to be that he was referring to the psychological impression that lower intervals are larger than the corresponding higher intervals.

[197] This characterization of the Alto as the voice with the most restricted range is contrary to the view of the musical theorists and composers of the time. See Dickreiter (1973), 186.

See Bk. II of Giovanni Maria Artusi of Bologna on the art of composition which is all about dissonances.[198]

casionally use major dissonances, in such a way that the dissonant note is one whole tone distant from the one which would make a consonance, yet that does not occur except for expressing and eliciting the most serious disturbances of the mind. However, the ordinary dissonance, mixed with pleasure and thus in a sense natural, is completed by a semitone. Again the cause of this, in order for the last to correspond with the first, is to be sought in the deepest foundations of geometry, in Axioms II and III of Chapter I, and in the first Book and the study of the fifteen-sided figure. For since the semitone is marked out by the numbers 15 and 16, and indeed there is a construction not only for the sixteen-sided figure but also for the fifteen-sided figure, we very nearly found that 15:16 defined a harmonic ratio, and equally also those derived from it, 15:8 and 15:4 and 15:2 and 15:32 and 15:64, and above all 15:1, that is to say those which the composers most frequently admit for forming legitimate dissonances. We had no reservation except that the construction of the fifteen-sided figure was not modeled immediately on the number of angles in the figure, like the rest, and so was not its own, but transferred from different figures, the triangle and pentagon, and so second hand and of a very remote degree; and that although an angle of the fifteen-sided figure was congruent with the others as far as filling a position in two dimensions was concerned, yet the whole figure did not admit congruence in respect of all its angles. Since, then, 15:16 and the allied ratios approach so closely to consonances in respect of the cause on which they are modeled, what is surprising about their being frequently mingled with the consonances in practice? However, on the other hand the tones 8:9 and 9:10 participate with one of their terms in each case in the nine-sided figure, which is completely inconstructible. Hence the dissonances introduced by them and the ratios allied to them (4:9 and 2:9 and 1:9 and 9:16 and 9:32, and similarly 9:5 and 18:5 and 36:5 and 9:20 and 9:40, and so on) are not pleasing but altogether rough, and much rougher than those formed by conflation from the tritone and the mutilated diapente and similar intervals. On these see Artusi. For they arise after the major and minor tones and the semitone, that is to say from unnatural combination of these, as elements. Yet always the departure from the pleasing is equivalent to the departure from nature.

The force of the fifteen-sided figure in the delight given by dissonances.

Bk. II, 29 & 30; Bk. I, 44.

Bk. I, 47.

Syncopation.

Third, their other name implies the other laws of dissonances, when they are called syncopated. This is a feature common to both dissonances and imperfect consonances; but we are now dealing with dissonances. For ordinarily their characteristic is that they use it like yeast or salt or vinegar in cooking; that is to say complete dishes are not made from them. Similarly in this case also whole tunes are not brim-

[198] G.M. Artusi, *Seconda parte dell'Arte del contraponto* (Venice, 1589). Kepler wrongly refers to this as Book II.

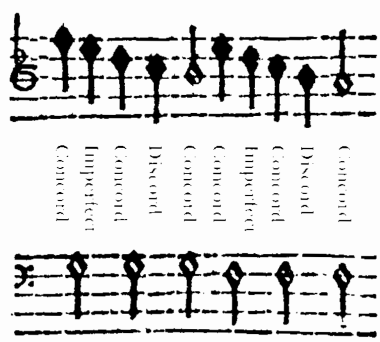
ming with them, except for great emphasis. No single part as it starts at the beginning of a measure is itself dissonant as it runs among the rest which are consonant; but furtively, and as if behind its back, it admits dissonance. In one position in the system, which it took up on the upbeat, and thus in beginning a particular note in the later and less important part of the measure with the agreement of the others, after they have finished, it delays longer and beyond the beginning of the following bar, in such a way that all the other parts, very often lower than it (that is to say, so that by its smallness and weakness and quickness, inasmuch as it is high, it is less obtrusive) in the general harmony at the beginning of the following measure take up a position which is dissonant with its position, before it has departed from its own position, and drive it from there, usually into a lower position, as if it were reluctant. It seems that it is also from this circumstance that the name of cadences was born for consonances and closes which resolve dissonances.

Cadences.

Furthermore they observe that the struggle of dissonances is ordinarily carried on not so often with a neighboring interval, and between seconds, as between sevenths, so that the upper part, driven on this basis from its own position, slips down a semitone or a tone to a sixth, or flies up to the octave which is to be indicated by the notes which have gone before.

There are also other very frequent dissonances, less the product of art, when a low part continues its note, staying in unison, but a high part sallies forth from the octave of the low note with rapid tinkling to some note which is consonant with it through a diapente or diatessaron, fluttering through all the intermediate melodic positions in such a way that it is in consonance and in dissonance with alternate notes. However, the earlier in order is always in consonance, and the dissonance is brief, inasmuch as it is transitory, and is with the note which is later in order, so that if a measure couples five notes of the Treble with one of the Bass, in double time, the first, third and fifth of the Treble will be consonant, the second and fourth will be dissonant, or in rather imperfect consonance. Like this:

Common dissonances.

Alternation of consonances.

III. Therefore the reason for this rarer combination is the same natural one as for the other very frequent and constant one, while the consonances vary. For just as in a simple tune the path of one individual part from the point of origin of the system, through intervals which are dissonant but melodic, tends towards positions which are harmonically tempered to the point of origin, and lingers in them, so tuneful part singing, generally starting from unison, by way of minor consonances, commonly called imperfect, or even of these dissonances, tends towards major and perfect consonances, and mostly (especially at the end) towards identical consonance.

Imperfect consonances.

IV. Again just as in simple ballads in those languages which are based on rhythm and on the length or shortness of syllables, in fact not much notice is taken of the short ones, but we direct the long syllables, and the ends of verses, to positions which are consonant with the first note, similarly in part singing there are consonances called minor and imperfect which we flit over, and there are also perfect consonances, towards which we tend, which are therefore called closes.

Closes.

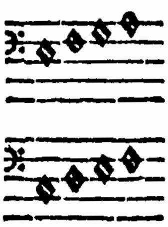
Perfect consonances not to be placed in succession.

V. Furthermore just as the actual individual consonances considered separately are pleasing on account of the fact that they are plainly not identical notes, but in a way figured and different notes, and take up a certain qualitative breadth in the hearing, drawing their nature from the regular plane figures from which they also take their origin, in the same way in this case the harmonious singing of parts, or the constant sequence of several harmonies, without any variety in them, ceases to be pleasing altogether.

Why two diapentes, and so forth, in succession.

From this explanation of the definition the reasons can be declared for the laws which polyphonists observe in fitting together their consonances. For the question can be asked, what is the natural reason why they believe it legitimate to arrange several natural thirds and sixths in succession, but do not admit several fourths and fifths and octaves and the like, from individual kinds in succession?

I reply that the reason is that thirds and sixths which are consecutive in pitch are mostly in actual fact different, one major and the other minor; and although sometimes two minor thirds follow each other very closely, yet they are consonances to a very small extent and imperfect. For let the first part sing *Ut, Re, Mi, Fa,* and the second at the same moments of time *Mi, Fa, Sol, La.* There are four consecutive thirds, of which the first, *Ut–Mi,* is major; the second, *Re–Fa,* and the third, *Mi–Sol,* are minor; and the fourth, *Fa–La,* is again major. On the contrary, all fourths are commonly taken to be intervals of the same size, as are fifths, and octaves much the most of all. Thus if several of these perfect consonances of the same kind were arranged in succession, differing in pitch, the melody would have no variety in consonances.



But another important reason prevents several fifths from being sung in order of pitch, which does not prevent octaves from being sung. Aristotle thought he should enquire about this in his *Problems.*[199] Why do they sing octaves in harmony (for example, males with females, men with boys), but do not sing fifths and fourths? The reason,

[199] Aristotle, *Problemata,* Book 19, problem 17, 918 b 30–40.

then, is this, that whereas two thirds make the consonance of diapente, and two octaves the consonance of Disdiapason, on the other hand two fifths make a ninth, and two fourths a seventh, which are dissonant intervals. Therefore in the former case the voice as it runs through octaves and thirds equally remains in the tone or mode, observing the same kind of octave; in the latter case if it runs through equal intervals, it changes the mode or tone noticeably, which is contrary to the definition.

VI. Certain remarks should also be made about the delight given by consonances: first, that in general large intervals express large disturbances of the mind (as in Orlande's "Where is Abel," and "Sad is my soul"); small high intervals, joy; and low small ones, sadness.

Second, since harmonized melody mostly finishes in identical consonance, so that, according to the hypothesis of this text, we encompass a large interval at the end, not only should the Bass go down, but also the Treble, against the nature of ending a melody, should go up, in imitation of Panpipes and the strings of the pandora. And then most of all the force of the semitone, explained in the previous chapter, shines out; that the natural and ordinary end of a tune likes at the close to go over a semitone rising, that is, in those tones which do not have a semitone next above the point of origin of their octave.

Why large intervals are more enjoyable.

Third, Galilei[200] thought it worth asking why the fifth above the octave sounds more pleasant that the simple fifth, and thus in consequence one harmony is always more pleasant than another. The reasons which he adduced are not worth examining, since they are obscure and uncertain. However, from my basic principles the reasons for this fact are very clearly apparent. For because by Axiom II of the first chapter the degrees of knowability by which the sides of the figures differ from each other, are also transplanted to the actual proportions, the progeny of the figures, it is indeed in conformity that the degrees by which the constructions of the part of the circle and of the remainder are distant from each other also differentiate between the pleasantness of the part and the remainder. Then those harmonies which arise immediately from the actual division of the circle, that is to say those which are between the whole and the part split off, are more perfect and more pleasant than those which are between the remainder and the whole, which are after all derived from the earlier ones, through the admixture of identical consonances from the bisection or doubling of strings. Thus 1:3 contains the first sort, and is therefore more pleasant than 2:3, and 1:6 than 5:6 and 1:5 than 4:5 and 1:4 than 3:4 and 2:5 than 3:5 and 3:8 than 5:8. For the former is a consonant part in its own right, not on account of the remainder; the latter, and thus more imperfect part is shown to be a consonant remainder, that is to say from its part's being allied or alien.

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[200] Galilei, *Dialogo*, 135–138.

The remaining rules of the art of polyphony I leave to the practitioners themselves to test or to supply reasons for. For me the arguments given up to this point are amply sufficient, both for illustrating the nature of melody, at least by revealing its primary general foundations in nature, and most of all for the speculations in Book V which follows. The theme of that book is the sole object which I intend in this whole work. For being an astronomer, just as I argue about the regular figures not so much geometrically (except where the geometry still seemed incomplete) as astronomically and metaphysically, so also I write about the ratios of melody not so much musically as geometrically, physically, and lastly as before astronomically and metaphysically. For just as I labor over the five regular solids in geometry, so also do I labor over the proportions and the whole harmonic panoply in music, in order to explain the reasons for the proportions of the celestial circles, and their eccentricities and their motions at the apsides. However, I do not in any way profess the art of harmonizing melody, which is the practice of music. Anyone who wants to, will do better to look for it in the books of Artusi, who was mentioned above, and those which Seth Calvisius[201] former friend of mine published on the art of composition. I name them not because I think them the best, but because I have seen no others.

Goal of this whole work.

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[201] Seth Calvisius, *Melopoeia seu melodiae condendae ratio* (Erfurt, 1592) and *Compendium musicae pro incipientibus* (Leipzig, 1594). Kepler's account, in this chapter, of principles of harmony such as the form of cadences and the proscription of consecutive fourths, fifths and octaves, seems to have been taken from Calvisius. See Dickreiter (1973), 187.

Political Digression on The Three Means.

On page 165, line 13, the following paragraph was omitted from its place, because in the process of printing the pages of the original had been carelessly distributed.

". . . (in other words 7:10.)

However, the authorities who accept into the state proportions of this kind under the heading of Harmonic should not be blamed on that account. For although few of the proportions thus formed define pure harmonies, and all the rest are foreign to harmonies, yet they all conform with one point in the definition, that they are a combination of both proportions, Arithmetic and Geometric. It is from

See the splendid passage in Bodin[202] on the state.[203]

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[202] Jean Bodin. *Les six livres de la république* (Paris, 1576). The author prepared a Latin translation with extensive revisions, *De republica libri sex* (Lyon, 1586). The edition cited in these notes is *Ioannis Bodini de republica libri sex*, edition altera, priore multo emendatior (Frankfurt, 1591). There is a contemporary English translation based on both the French and Latin versions by Richard Knolles, entitled, *The six bookes of a commonweale* (London, 1606). A facsimile of this edition, with corrections and introduction by Kenneth Douglas McRae (Cambridge, Mass., 1962), includes a bibliography of Bodin's work, pp. A78–A86.

Bodin applies mathematical proportions to political theory in the last chapter of his work, Book VI, Chapter 6. According to Kepler, Bodin's application of proportions to politics is flawed because Bodin believes harmonic proportions to be combinations of arithmetic and geometric proportions. He agrees with Bodin that arithmetic and geometric proportions relate to the administration of justice. In Kepler's view, however, the basis of the higher aspects of government ensuring the welfare and integrity of the state, is related to harmonic proportions, and these, he demonstrates in opposition to Bodin, are independent of arithmetic and geometric proportions.

[203] The word *respublica*, which has been translated here as "state," suggests that the source of ideas is Plato's *Republic*. However, Bodin's account of harmony in the state, which fascinated Kepler in spite of crucial differences, is widely different from Plato's. In Plato, the ideal state is studied as an enlargement of an individual person, so that the true nature of justice, or righteousness, can be perceived. Plato, or rather Socrates as a character in Plato's dialogue, argues that, in the ideal state harmony occurs when each of the three classes performs its own function—the philosopher-kings rule, the auxiliaries or soldiers defend and the working classes work. If one class usurps the function of another, then an inferior state results. Hence states are classified, in descending order of inferiority to the ideal state, which is aristocracy, as timocracy, in which the soldier class is most honored, oligarchy, in which the rich are in control, democracy, in which the ordinary people govern the state, and worst of all, tyranny, in which a single individual seizes power. Types of individual human being correspond with the five types of state. Living in what Plato would clearly have viewed as an inferior type of state, Kepler could hardly have been foolhardy enough to adopt this view even if he had wanted to. Although in principle Plato believed that geometry was extremely important and part of the education of future philosophers, mathematics plays very little part in his own surviving writing. Only one or two incidental uses are made of the subject in the Republic, and those are not altogether serious.

this combination that they derive their suitability for the state. Correspondingly, however, if in the state there is any power in the harmonic proportions, as harmonic, neither will it be possible to disdain our harmonic means, which I state by a looser definition as follows:

That it is anything which comes between two concordant notes and is itself in concord with both of them.

Therefore in the harmonic divisions, etc. . . ."

What a Harmonic Mean really is.

Left out until now. Now when I had mentioned Bodin in the margin, and the third Book was already finished and the paragraph above had been added, I had thought that enough attention had been given to mathematical speculations. Having finished this thorny material, I decided to transcribe from Bodin himself the main heads of this political dissertation, shaping the words and arrangement as far as possible from my understanding of that passage. That was both to shed some light on a passage which by itself is obscure, inasmuch as Bodin did not sufficiently fortify himself with mathematical learning for such speculation, and to lighten the tedium of dour mathematical demonstrations, of which the whole book consists, by the interpolation of some enjoyable popular material, and to display a foretaste of its considerable usefulness in understanding the State. However, I shall begin with a mathematical explanation of the matter which is rather extended and more suited to the understanding of the reader who is not a mathematician.

When to some numbers, regardless of their size, equal amounts are added, then the proportion is arithmetic. Thus

For 6 is greater than 3 by the same amount (that is to say by as many unities) as 12 is greater than 9.

	3.	9.	5.	10.	17.	38.
Add	3.	3.	3.	3.	3.	3.
	6.	12.	8.	13.	20.	41.

The proportion in this example is discontinuous. Then a continuous proportion or arithmetic progression is when, starting from any number, equal amounts are added to it continuously.

Then since between 3, 6. 9 and 12, and also between 38, 41, 44 and 47 there is a continuous arithmetic progression, of three such numbers placed in succession the middle one is called the arithmetic mean. Thus between 6 and 12 the mean is 9; between 38 and 44 the arithmetic mean is 41.

Such as	3	or	38
	3		3
	6		41
	3		3
	9		44
	3		3
	12		47

However when to some numbers are added numbers similar in relation to their size, then the proportion is geometric.

For just as to 3 is added three times itself, 9, so to 9 is added three times itself, 27,

Such as	3.	9.	5.	10.	17.	38.
	9.	27.	15.	30.	51.	114.
	12.	36.	20.	40.	68.	152.

which is as much greater than 9 as 9 is than 3 and as 15 is than 5, and so on. Then as 3 is to 12, so 9 is to 36. And as 3 is to 9, so 12 is to 36.

Again the proportion in this example is discontinuous. Then a continuous Geometric proportion or progression is when starting from any number either a similar fraction or a similar multiple is perpetually added to it.

To the initial number is added three times itself in the two first examples, half in the third; and again to the number thence produced is added the same multiple or fraction. Then as 8 is to 12, so furthermore is 12 to 18 in this case, and so also is 18 to 27. Here 12 is the geometric mean between 8 and 18; and similarly 18 is the geometric or proportional mean between 12 and 27, and so on.

Such as	3	10	8
	9	30	4
	12	40	12
	36	120	6
	48	160	18
	144	480	9
	192	640	27

These things must first be grasped in order to understand what harmonic proportion is. For it is defined by Bodin[204] as that in which *the ratios of equal and similar are combined in a proper manner*, that is the earlier arithmetic ratios and the later geometrical ratios.

Furthermore this definition is not true. 1. For there are many ways also in the case of numbers which are not harmonic, in which the ratios of equal and similar can be combined in a proper manner; but that is not sufficient for the proportion to be harmonic. For there must be added other characteristics of the numbers, to be looked for in Chapters I, II, and III. 2. There are on the other hand many harmonic couplings of numbers in which the ratios of neither equal nor similar are contained. 3. Furthermore there are harmonic proportions in which there is a simple geometric proportion, such as 1, 2, 4. Although they are identical, they do not generate on their own any particular pleasure, to such an extent that Bodin says they do not produce by themselves parts which are in harmony, meaning figured. There are on the other hand other harmonic proportions in which there is a simple arithmetic proportion, such as 1, 2, 3, and similarly 2, 3, 4, which even Bodin[205] forgetting himself recognizes as harmonic, although they have no admixture of the geometric analogue. Such are 3, 4, 5, and 4, 5, 6, and 1, 3, 5, and 2, 5, 8, etc. These last Bodin falsely says do not produce harmonic parts, rebelling on the authority of the ancients against the sense of hearing.

However, there is one assertion which is true up to a point about the proportion which the ancients with misplaced conviction called harmonic. I say, "up to a point"; in other words if instead of *ratios of equals* you understand arithmetic series, transformed into other numbers in which there is not shown the actual equality of differences which there was in the first three. For the ancients very properly defined the harmonic mean as *that which makes the proportion of the differences between three numbers the same as that of the outer numbers*, as in the case of 3, 4, and 6. Whatever three such numbers are, whether they are truly harmonic or not, it is always true of them that the ratios of equal

[204] Bodin, *De republica* (1591), 1170–1171.
[205] Ibid., 1216.

(or rather arithmetic) and of similar (or geometric) are combined in one definite way. Now that way is as follows. Given two outer numbers, such as 2 and 5, the arithmetic mean is found. So that no fraction is involved, in this case take twice the two outer numbers, 4 and 10. The arithmetic mean of these is 7. Now in the case of such an arithmetic series, 4, 7, and 10, two ratios with the equal differences, 3 and 3, are always set up, one greater, between the smaller terms 4 and 7, and the other smaller, between the greater terms 7 and 10. If anyone now wants to produce the harmonic series from this arithmetic series, and to mix it with something of the geometric series, he should put the smaller ratio in the first position, and the greater in the second position, like this,* and seek the number which is to 10 as 4 is to 7. The result will be three numbers like this: 7, 10, $17\frac{1}{2}$, or in integers which have no common factor 14, 20 and 35. This trio has some admixture of the arithmetic progression, though the equality which it showed does not remain. For each proportion of the arithmetic series between 4, 7, and 10 has been brought over into it, with only the location changed; for as the first two terms were in the former case, that is to say 4:7, so in this latter case the second two terms are, 20:35. And as the second two terms were in the former case, 7:10, so in the latter case the first two terms are, 14:20. On the other hand there is also an admixture in it of the geometric series. For the smaller ratio, 7:10, has been applied to the larger terms, 14 and 20; the larger ratio to the larger terms, 20 and 35. And it finally turns out that as the smallest term 14 is to the greatest, 35, so the smaller difference 6 is to the greater 15. All these things smack of some family resemblance to geometric proportion, because in it the smallest term is to the mean as the mean term is to the greatest.

*7:10, 4:7.

These points have to be grasped from mathematics; and Bodin[206] does not seem to have perceived them sufficiently when he proposes harmonic proportion in the five numbers 3, 4, 6, 8, and 12, and declares of them that they show equal and similar spaces which are in a way tempered. For since he saw that the differences were 1, 2, 2, and 4, he recognized equality as appearing in 2 and 2, similarity in 1, 2, and 4. But in the definition of the ancients three are sufficient; and there is nothing in it about tempering the equality which appears with similarity. For in 3, 4, and 6, no equality appears between the differences.

It is now worth seeing how Bodin, who is very profuse on this theme, applies these points to morals and politics, in different ways. It often happens that instead of harmonic series, either those of the ancients or true ones, he drags in mere combinations of arithmetic and geometrical proportions, depending on his faulty definition.[207]

[206] Ibid., 1170–1171.

[207] Bodin had been of the view that the approximate middle between an arithmetic and a geometric proportion could be called a harmonic proportion.

just as this had to be stated straight away at the beginning, so it will ave to be repeated rather often in what follows.

I. There are three forms of constitution, the popular, the aristo-ratic, and the royal.[208] He compares the popular to the arithmetic roportion, the aristocratic to the geometric proportion, and the royal the harmonic. For just as in the arithmetic the increments of all e numbers, both great and small, are equal, so in the state the people ants everybody's burdens, and benefits, and honors, and offices, to e equal, and is unwilling to tolerate any respect of persons. For in-ance it wants everyone to have the right of hunting, whether they e noble or ignoble, or rich or poor. But if there is anything which oes not permit division among many, the people want lots to be drawn r it, because a lottery is blind, seeing no distinction between noble d ignoble, rich and poor, deserving and undeserving, the devotedly rtuous and the vicious, the clever and the stupid. For it is then that e thinks himself equal with the rest, when he draws lots with them r such things, whether the lottery brings him good or evil. Then stead of a lottery there can be other means also of making alloca-ons, as when the choice is deputed to someone to whom it makes difference, or who is believed to be no respecter of persons. Thus Rome the consulate was common to all of a certain station[209] and was legitimate to seek it from the people, but not by largesse, so at the people should not be corrupted, and the rich should not be eferred to the less well off; though in that case by tacit agreement tue is taken into account, and judgment of that and of the deserv-g is deputed to the people.

On the other hand just as in geometry the differences in the num-rs are accommodated to the numbers themselves, so that a large mber has a large excess, a small one a small excess, similarly in

[208] Here Kepler is clearly following Bodin's simplified version of Plato's classifica- or perhaps of Aristotle's. Aristotle, who is more empirical and flexible than his her Plato in this area, saw kingship, aristocracy, and timocracy or polity as the e subdivisions of the good kind of state, corresponding with their corrupt forms, nny, oligarchy, and democracy, the three subdivisions of the bad kind of state. Aristotle, *Politics*, Book III and *Nichomachean Ethics*, Book VIII. Elsewhere, in the *ve*, Aristotle distinguishes four forms—democracy, oligarchy, aristocracy, and nny. That classification is, of course, for a different purpose.

[209] In the Roman Republic the two chief magistrates were the consuls, who were ted annually. In the early days of the Republic after the expulsion of the kings, the upper class of patricians were allowed to hold the magistracies, and the t the citizens, the plebeians, were excluded. After a long struggle, the plebeians won the right to hold magistracies. However, the result was simply that a new er class, called the nobles and including families that had originally been plebe- was formed. It was very rare for a "new man," that is a candidate who was not mber of a noble family which had previously held the office, to achieve election at office. Kepler is aware of both distinctions—that between patricians and plebe- and that between nobles and others—and may here be referring to either or both.

the aristocratic republic distinctions are made between persons, and also between burdens, rewards, offices, and duties: the outstanding ones are reserved for the aristocrats, the rest are left for the people. Here it is necessary for arithmetic proportion also to be admitted separately within individual classes: for all those who are of the people will draw lots for what belongs to the people, all the aristocrats for what belongs to the aristocracy. For unless that is done there will be endless degrees of aristocrats even among the people, down to the lowest dregs of the people; and endless degrees also among the aristocracy, up to the chief man in the republic. On that basis it will not be a republic, but the royal kind of state.

However, though the royal condition is indeed accommodated very largely to the geometrical proportion, because all laws are reserved to the king's majesty, just as he himself takes precedence over all others either by his noble birth or by arms or by his excellences, yet the basis of government in this condition can most properly be tempered by both kinds of proportion. For a single king is the arbiter of all things, not by blind impulse, like a lottery, but on the bases of excellence, merits, order, and degrees, as far as is permitted, distributing everything among the aristocrats and the people, and executing all aspects of justice in distribution and exchange. This union of both proportions is sufficient for Bodin to establish a harmonic proportion; but accordingly he relates all policies not so much to individual orders or men, as to the whole body of the state, and its safety and mutual love and peace, exactly as if in the case of numbers the proportions are considerably diverted from equality and from similarity, so much so that those properties are destroyed if need be, and related to the common harmony of all. This is how he uses my harmonic means.

The judgment of the young Cyrus.

II. Bodin puts forward an example from the boyhood of Cyrus[210] by which all three kinds of proportion can be explained. Cyrus as a boy caught sight of a tall man wearing a short tunic, and a dwarf nearby with a roomy tunic and decided that they ought to exchange tunics so that each obtained what was beneficial for him. His master decreed that each should be left his own. But if the tall man had been instructed to pay the dwarf some money, and in that way the exchange eventually came about, the master and the disciple would have been very well reconciled. Here Cyrus urged the geometric proportion, adjusting the clothes to the bodies, his master the arithmetic, with an eye to each one's possessions; but a third party would have looked to both at the same time, both their bodily needs, and keeping their

[210] Cyrus the Great, founder of the Persian empire, conquered Lydia, the Greek cities of Asia Minor, and Babylon, and died in 529 BC. The story quoted here is derived from Xenophon, *Cyropaedia*, I. 3, and referred to by Bodin, *De republica* (1591), 1169. Both Bodin and Kepler take liberties with the story told by Xenophon.

resources intact. Thus he would have asserted harmonic proportion in Bodin's sense, inasmuch as it would have been a mixture of both the previous two, but harmonic also in my sense, because he would have no hesitation in taking away from the one as much of his clothing as was superfluous, and from the other money, consulting the interests of both; for the common benefit of both is compared with the pleasantness of singing in harmony. However, I said that it happens sometimes that geometrical proportion is also harmonic, such as 1, 2, 4, just as in this case it is beneficial for the tall man to have the long garment; whereas sometimes arithmetic proportion is also harmonic, such as 2, 3, and 4, just as in this case it is beneficial for the dwarf who possesses the long garment to keep in fact his own, but exchanged into money, which he can put to more proper use than the surplus of his garment.

III. Since justice embraces all bases of government, he adduces from Aristotle two aspects of justice, of exchange which functions in arithmetic equality, and of distribution in geometrical similarity.[211] Kinds of justice. He himself commends a third kind of justice, formed by conflation of the two earlier kinds, which intends both plenty of goods to be sold to certain persons at a cheaper price than the others, and great things not always to be allotted to the great; and thus for each kind of justice to be offended up to a point, if the safety of all demands it, or if public benefit is derived from it. The proportion is not well formed by conflation of the other two if it destroys both. Therefore this application is better fitted for my harmonic means. Thus in the numbers 2, 3, and 5 there is neither arithmetic equality of the differences, nor a geometric progression, but there is a harmonic progression between them.

With this in view at the end of the book he fits in the poets, who imagine three daughters of Themis (who is Justice), Εὐνομία, ἐπιείκεια, εἰρήνη, Laws, Equity, and Peace, as if they are tutelary goddesses of the three proportions, arithmetic, geometric, and harmonic.

IV. The laws of marriages, the chief bond of the state, supply a brilliant example of the three proportions. If patrician men are or-

[211] In Book V of the *Nichomachean Ethics* Aristotle distinguishes first between general justice and particular justice, and within the class of particular justice, between distributive justice and rectificatory justice. Distributive justice is the justice shown by the whole state in distributing benefits among its members not equally but in proportion to their merits. Rectificatory justice is the justice which the state tries to maintain between its members; but the justice of exchange, which is a special kind of rectificatory justice, requires that in trade each partner should give the other the exact equivalent, as measured by value in money, of what he receives. Aristotle relates distributive justice to geometrical proportion in *Nichomachean Ethics*, V, iii, 12 (1131 b 12) and rectificatory justice to arithmetical proportion in V, iv, 3 (1132 a 1). In short, Bodin has selected to suit his own purpose just two points from Aristotle's much more elaborate discussion. Kepler has simply followed Bodin.

dered to marry patrician women, plebeian men plebeian women, there is a geometrical similarity.[212] But if all without distinction have the right either to try to obtain marriages by lot, or to compete for them to the best of your ability either by looks or by wealth or by merit, without regard to birth, and no rank is forbidden, that will be arithmetic equality. But in the former case the minds of the citizens are torn apart into factions, in the latter case the orders of citizens are confused together. Either is destructive to the state. Bodin therefore recommends that the less well off patricians should sometimes be allowed wealthy plebeian marriages, and rich plebeians to marry needy patricians.[213] For this is beneficial to both orders, to the nobles indeed for them to safeguard the position which they hold with the help of increased wealth, and for women of their order to be married; but to the plebs so that they will work for excellence, when the path to distinctions is opened to them; and lastly to the state itself, for the orders to embrace each other with mutual love.

Marriage laws.

Again he has extinguished the geometrical analogy to bring about the harmonic one, but he has not reduced it to the mere equality of an arithmetic progression. Thus if the geometrical series, 1, 3, 9 is extinguished, and 1, 3, 8 introduced, the loss to the patricians on the one hand is as great as the retreat from 1, 3, and 9 on the other; and the gain to the plebs in the former case is as great as the amount by which in the case of 1, 3, and 8, the differences between which are 2 and 5 (whereas previously they were 2 and 6), we have advanced towards the arithmetic progression 1, 3, and 5 (because the differences between them are equal, 2 and 2); so that in the numbers indeed concord would be established between 1, 3, and 8, and in the state concordance between patricians and plebs, though without losing the distinction between the orders. This change of 1:9 into 1:8 will also occur in Book V in the actual heavens.

V. There is an absurd arithmetic equality at banquets if everybody is seated indiscriminately, with no account taken of sex, condition, or age. On the other hand mere geometric similarity is insipid. For if the learned are only put next to the learned, what good will they do to the unenlightened? If women only next to women, what pleasure will there be? If the rowdy next to the rowdy, who will instil good behavior into them? But if you admit neither blind equality, nor peevish similarity, but mix them both, though with moderation and with reasoned judgment, the proportion will be harmonic. For you will bring it about that the old rejoice to see the young, the men to see the women, the young are ruled by the wisdom of the old, the women by the authority of the men, the sociable stimulate the unsociable, and on the other hand are also respectful, and do not aban-

Law of banquets.

212 For the distinction between patricians and plebeians, see note 209 above.

213 Bodin, *De republica* (1591), 1172.

don themselves to buffoonery. Again this is not a combination of intact kinds, but to a certain extent an infringement of them, to set up a harmonic proportion.

VI. It belongs to arithmetic equality that in ancient times the patricians used to watch the games sitting along with the plebs indiscriminately; and to geometric proportion that afterwards the patricians were segregated from the plebs.[214] That indeed is the position if you consider the games alone. But if you keep all the benefits to the republic in a single mental vista, since a great many of them would have been preserved for the patricians in geometrical ratio if the people had been segregated, this placing of both orders on an equal footing at the games belonged to that tempering, which was harmonic in Bodin's sense, of the whole pattern of government, inasmuch as in it not only were the plebeians cajoled, as boys often are by trivialities, and attracted to loving the superior orders, but also nothing was lost to the dignity of the orders. It also squares with my means, insofar as the proportion is not retained in order that concord may be retained. Previously there was a loss to the relationship in certain persons belonging to the orders, in the present case in certain privileges.

Sitting together at games.

VII. Friendships are given life by harmonic tempering. For what concord is to proportion, that love, which is the foundation of friendship, is to the whole compass of human life. But if mere equality of offices is brought in, according to an arithmetic law, there will be no friendship except between equals; and if between unequals there is a mere detailed geometrical similarity of offices, there will be friendship on neither side, but in the former case a perpetual dealing and trafficking in offices, for their own benefit, in the latter case an unavoidable association of patron and client, no freedom to demonstrate love, and nothing spontaneous. Although friendship cannot survive frequent injustices, yet it rejects laws, and refers everything to the sound and sober judgment of love, dispensing now equality, now proportionality, and when neither, always dispensing what seems in the immediate situation to make for the preservation of love, which is also goaded on, as harmony is by discords, and as fire is by an iron poker, by a few injustices and renews its strength by free forgiveness of them.

Friendship.

VIII. Arithmetic equality is excellently compared by Bodin to the iron rule of Polycletus, which could be broken sooner than bent; and geometrical similarity to the leaden rule of Lesbos, which was bent to fit all angles.[215] He himself hints at harmonic proportion, on his

The rule of Lesbos.

[214] Ibid., 1173–1174.

[215] Ibid., 1175. Polycletus or Polycleitus of Argos, a Greek sculptor of the fifth century BC, tried to express arithmetic harmony in the proportions of the male body depicted in his statues, and wrote a lost book, *The canon*, which was criticized for prescribing rules for bodily proportions that were too rigid. His rule was thus meta-

own definition, by the wooden rule, which bends indeed, but returns from its bent position.

Laws and Justice.

IX. In the government of every kind of state, rigor in the laws and the office of judge are compared to arithmetic equality, in which not only is each allotted his own to the last farthing, but also penalties are imposed on delinquents equally, without respect of persons, and in which judges are bound by the laws, and by what is alleged and proved, to judge according to the laws even if they seem unjust. On the other hand justice, and the mere judgment of the magistrate, who is, however, a good man and acts according to conscience, takes on the nature of the geometrical relationship. But Bodin in his example of harmonic tempering inserts an intermediate duty, either for the higher courts, or all of them or the highest magistrate. They are not allowed to act on a mere legal judgment, but are permitted to bend, not break, the laws by taking account of the circumstances, and to interpret them in their judgments in accordance with justice, though in such a way that nothing is lost to the law by individual judgments of the courts. He wants the laws to be considered like his yardstick, so that they yield indeed to bending, but spring back at once. Nevertheless in this passage Bodin by his example has demonstrated a truth which he had asserted previously about lawyers, that they have rather little perception of this mathematics on account of its obscurity. For his argument about numbers does not square with the rule of our discipline, when he says that it cannot come about that a geometrical proportion which holds between four terms is destroyed by any transposition of the terms; and he declares that there is the same relationship between 6, 3, 4, and 2 as there is between 3, 2, 4, and 6.[216] In no way is 3 to 2 as 4 to 6. On the contrary he thinks that a harmonic proportion is upset by transposition. Certainly he wins if he recognizes no proportion between two terms on their own.

What I mean by proportion.

For this is some people's subtle way of speaking, combined with the utmost obscurity, that what the Greeks call λόγος [ratio] they want to call in Latin *ratio*, but *proportio* what the Greeks call Ἀναλογία. For my part I should like to be able to imitate that, as I remember I did previously; but λόγος is never taken by the Greeks in the common usage of speech instead of the word αἴτιον (cause), whereas in Latin *ratio* very often means "cause" or "means." We must therefore keep the custom introduced by the barbarian commentators on the Greek Elements,[217] and adopt the word "proportion" both for λόγος and for Ἀναλογία, which I do frequently in the whole of this book.

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phorically of iron. The rule of the Lesbian architects, however, was said by Aristotle to be made of lead because it was therefore flexible and could be worked to take the shape of the stone. *Nichomachean Ethics*, V, 10, 7.

[216] Bodin, *De republica* (1591), 1176.

[217] The reference is to the various scholia on the *Elements* of Euclid which were written in the later Roman Empire.

The distinction has now been made, then, between proportion and concord; and the latter is like a sort of quality of the former, and the former subject to the latter. Concord indeed is in the proportions primarily and in its own right, insofar as the proportions are between two terms, not insofar as there is some continuity of proportion between several terms. Also two terms of a proportion are concordant not by reason of position, in which there is an earlier and a later one, but insofar as two strings are struck at the same time; and this concordance is not changed when the position of the characters is changed in writing, or reversed on the lute. Indeed a proportion between three or more terms acquires concordance secondarily, on account of the individual pairs of terms, in which although if the position is changed the relationship can be changed, namely that which is mingled with them from geometrical proportion, yet the harmonic nature cannot be changed at the same time.[218] Bodin argues that these four elements—law, the action of the law, justice, and the magistrates—are comparable with four numbers linked in harmonic proportion; and just as if the numbers are transposed the consonance disappears, so if official proceedings and prosecutions are considered more important than the laws, or the magistrate superior to justice, the harmony of the state is disturbed. This argument in fact must not on any consideration be related to what is harmonic in the proportions, but simply to the nature of the proportions themselves, in virtue of being geometric or arithmetic, that is, to that kind of proportion which, being tempered from both, the ancients with a certain unreliable conviction called harmonic. For that is common to all the relationships whether simple or mixed, contrary to Bodin's statement, so that they are either changed into other kinds when their terms are transposed or flatly destroyed. But I return to this point.

If the magistrate has unfettered authority to govern without laws, and neither observes natural justice nor plays the part of a good and blameless man, he will not be a magistrate but a tyrant. And since the number of good leaders in a city is extremely small, it is not sensible for the state to expose itself to this danger every time a new magistrate is appointed. Indeed even if it gets a good magistrate sometimes, yet that situation cannot be permanent; but it is not healthy for the state because, however good its regent may be, he will not always be wise enough to see what is just without the yardstick of the laws, and while he himself may see and follow what is right, the citizens will unjustly blame even the best, unless there are laws by which he can safeguard his deeds against the people. The citizens must be educated in the discipline of the laws, and deterred from misdeeds by reflecting on the penalties which are prescribed by the laws. Therefore the state cannot do without laws.

The necessity for laws.

[218] Bodin, *De republica* (1591), 1176.

On the other hand the laws, in the Greek term Νόμοι, signify the distributive aspect of justice, which in the popular type of state follows arithmetic equality. But if everything is examined according to the rigor of the laws, injury will be caused to a great many. For laws are based on facts, stipulated without variation for circumstances; whereas individual facts depend on the circumstances, by which they are either mitigated or aggravated; and since they vary, they can never be covered by any laws. Therefore neither alone by itself safeguards the state; but Bodin tells us that they should be mixed, so that there should both be laws, but few of them, because a multitude of laws is a very fruitful soil for litigation, and also many things which are reserved for the judgment of the magistrate—not only individual facts, but also several complete classes of facts, and certain other legal matters, chiefly the extension and remission of penalties, the interpretation of the actual laws, and their application to the circumstances of the facts. Here it is a question not of the three forms of state, but of three forms of government of each one: the basis of pure justice, unfettered by the laws, is an aristocratic basis of government, even in the popular and in the royal kind. The restrictions of laws make the government popular even in the royal kind. But if there were to be laws, they would however be modified in relation to time and place according to the judgment of the governor, established by the highest power: that governor, though he may be appointed by the people, exercises a kind of rule. Such very often are the heads of provinces. He, then, observes the laws, or judges justly, as long as the safety of the state allows it; otherwise, he does not hesitate to deviate from them. Thus if one were to admit kinds of proportion which make harmony, such as 2:3:4 or 1:2:4, he would disturb those which make dissonance, and change them into consonant ones, for instance 1:3:9 into 1:3:8 and 3:5:7 into 3:5:8.

And for justice.

X. In the case of those laws which are about monetary fines, as far as possible indeed arithmetic equality is preserved, so that whoever has offended pays the fine established by the laws, whether he be rich or poor. However, we can see where there is a geometrical similarity, as if a fine is imposed on each in proportion to his standing, or if someone who goes to law frivolously loses a certain part of his assessed costs. But if some larger fine is specified among all the wealthy, and a lighter one among the badly off, and that is exacted equally from individual members of each order, that will be a tempering of the proportion which Bodin calls harmonic. The kinds of proportion are not changed in this case, but combined, or rather distributed among different categories, though belonging to the same whole.

Fines.

XI. Laws about clothes have much of the geometric proportion: so that in proportion to the degree of dignity at which each man stands, he is permitted more costly ornament on his clothes; here arithmetic equality is intolerable.[219] However, because it is not only the difference between the orders which calls for distinctions of dress, but here some account must be taken of riches, and some of merit, and because rules for all orders cannot easily be proclaimed in a single law, some tempering of both proportions is in part customarily inserted in the actual laws, and in part allowed to the judgment of the regents. This tempering Bodin considers harmonic, because he defines harmonies by the combination of two kinds of proportion. To me it is not harmonic, since another proportion is the aim, and would be perfectly expressed if it could be. Sumptuary laws.

XII. Almost the same must be said about those laws which are about punishments and exempt illustrious persons from the danger of execution, and nobles from hanging, and change the kind of punishments, in the manner accepted by most nations. For the accused, in losing the position which he had, is more heavily punished by this disgrace (observing the geometric law) than if someone who had no position was publicly flogged. Thus there can be a certain equality in dissimilar punishments, and a harmonic distribution in Bodin's sense, as he here rejects Aristotle, who had taught that in rewards indeed the geometric proportion should be preserved, but in punishments the arithmetic.[220] Punishments.

XIII. There is also another proportion with the same tempering to be observed in penal laws, affecting not the offender but the honor and retribution of the person wronged. For the punishment is heavier for killing a Head of State, or one of the College of Electors,[221] than a peasant; heavier for killing a free man than a slave. For although this variety must be related to geometric similarity between the person and retribution for the injury inflicted on him, yet it is rare. On the other hand there is very often arithmetic equality in punishments for homicide, especially among Christians, according to the divine law by which man is reckoned equal to man, and blood is repaid by blood. Therefore this combination of the two kinds of system belongs to what Bodin calls the harmonic, especially the fact that in the divine law all murderers are punished by death with arithmetic equality, but Reparation.

[219] Sumptuary laws regulating the expenditure of individual citizens were known in ancient Greece, at Rome under Cato the Censor, and in various European countries during the later Middle Ages and early modern times. Regulation of expenditure on dress according to social class was enacted, for example, by Philip IV of France in 1294, by Edward III of England in 1363, and by Edward IV of England in 1463. Bodin presumably referred to the French legislation.

[220] See note 211 above. Cf. *De republica* (1591), 1191–1192, where Bodin explicitly rejects this teaching of Aristotle.

[221] In Kepler's time and until its fall in 1806, the emperor of the Holy Roman Empire was elected by a group of rulers of states within it. At this date they were seven in number: the archbishops of Mainz, Cologne, and Trier, the King of Bohemia, the Count Palatine and the Margrave of Brandenburg.

the kind of death to be inflicted is within the power of the judge, in geometric correspondence with the dissimilar facts and variety of circumstances, according to Bodin's account, drawn I think from the teaching of the rabbis.

This inequality in punishments is due not so much to the individual persons injured as to the safety of the whole republic which permits all enemies of the fatherland to be killed with impunity, and at the same time safeguards the well-being of all the citizens in the security of the leader and the aristocracy, and in the preservation of public tranquility. However it has often been said that the public good has a certain correspondence with the way in which singing in harmonic parts is pleasing. If it is observed, this one supreme law, the mother of all laws—that anything on which the safety of the state depends is ordered to be sacred and lawful—is now at once consistent in respect of similarity with harmonic ratios, described also according to my own views, even if that law contains nothing further similar either to geometric or arithmetic proportions. For these numbers, 15, 20, 24, and 30, which contain the sweetest harmony, do not observe either one proportion or the other either, for their differences, 5, 4, and 6, are neither equal nor similar in proportion to the proportions of the terms, but are not in the same order of size as they are.

XIV. Bodin adduces another example, closer to the discipline of geometry, in the interpretation of the laws on fines, that the monetary fines which were established in antiquity are reckoned in proportion to present public wealth or poverty. This arithmetic equality at all times is intolerable; and yet geometrical proportion is not easy to observe in all cases. Thus in accordance with his wisdom, and in accordance with the circumstances of the case, the judge will strive for the harmonic mean, taking care neither to seem to have violated the laws nor to oppress the poor by malicious interpretation of the laws and give the wealthy and more powerful license for wrongdoing. Furthermore in zeal for the same harmony he will sometimes change the monetary penalty, which would be despised, to a physical one.

The reckoning of a fine.

XV. Bodin approves of the law which punishes thieves by hanging to the extent that it depends on the argument of preventing an increase in thieves, so that crimes which were going to become frequent should be punished more heavily, which has a flavor of geometric proportion. He criticizes the same law on account of arithmetic equality, on the ground that the very frequent prosecution of theft everywhere in town and country generally has plebeian judges, who in hurrying all equally to punishment do not understand that they are inflicting a very unjust inequality of penalties, when they impose the same punishment for crimes which are very unequal in the variety of their circumstances. The judgment of many criminals is unreliable, which ought to make them have a care for themselves more because of the misfortune than because of the greatness of the crime.

The punishment for theft.

It is opportunity, hunger, devotion to their children, confidence in going undetected, belief in other people's wealth, which makes them thieves, since their natural judgment makes light of the crime, compared with other things, which even divine law, which punishes theft only by fourfold restitution, seems to confirm. If, however, it is just and blessed that the part of mankind which, born in deceit and darkness, adopts the character of mice, burrows through walls, upsets the public security not only of fortunes but also of life, should be weakened more for the sake of the state, like other beasts of prey, certainly respect for the state, as has often been said, will correspond with singing in harmony, and this question should not be debated without the most accurate harmonic tempering, in case, while the interests of the republic are consulted, many who can offer hope of good citizenship in the future should have life heedlessly torn from them because of this kind of charge.

XVI. It also belongs to geometrical justice in the imposition of punishments that the penalty is more lenient for a boy than for an adult, for a young man than for an old man, for a woman than for a man. Here nothing is more unjust than perpetual arithmetic equality: however, if it is removed from the sphere of the judge's wisdom, it must be adopted in accordance with the immediate situation, and harmonic proportion will be disturbed none the less.

Punishment to be extenuated.

XVII. Bodin demonstrates harmonic proportion in those laws which avenge the first commission of a misdeed equally with a light fine in all cases, as if to serve as a warning, which afterwards punish second offenders more severely, and those who offend a third time still more severely, and which finally sentence them to death. For the arithmetic equality of the first occasion is mixed with geometric proportion in the more serious punishment for the repeated and therefore more serious crime. This combination of the two into one seemed to Bodin to constitute harmonic proportion.

Punishment of the recidivist.

XVIII. In general as punishments are apportioned, whether by the laws or by the judge's discretion, in accordance with the diversity and magnitude of the charges, and therefore the universal rule of retribution must be attributed to the geometric relationship, and certain legislators exact equal retribution for misdeeds which are unequal in kind, this arithmetic ratio has no place among men, and it is not right that it should be rebuked by our complaints in God, whose creation we all are. But when legislators for the sake of the state make certain charges which are lighter in themselves equal to the most serious in the infliction of punishment, and when in the rule of retribution not a tooth for a tooth, but in place of a tooth an assessed price is paid in recompense, on which there can be an agreement among the litigants, and on him who has torn out one of my two eyes, as he was one-eyed himself, downright blindness is not inflicted by tearing out his one, that for Bodin is an instance of harmonic tempering. For

Punishment as retribution.

in that way not only do individual misdeeds have arithmetically equal punishments, but also different ones are differentiated by their punishments.

Punishment according to intention.

XIX. Those who punish inclinations of the mind with no lighter a punishment than if action follows the wish, and who punish an act from which intent was absent, commit no small fault in entrenching plebeian arithmetic equality, in the former case in dissimilar inclinations of the mind, in the latter case in dissimilar transgressions. He who distinguishes in the former case between various forms of inclination, so that the punishment for the crime goes to the man who has done everything which his mind after reflection can bring to realization, whether he has carried it through or whether he has missed his aim, but does not go to the man who has in fact watched for opportunities, but has not arrived at the actual deed, the guilty imagining of which could deter him, he observes geometric justice. No less does he who punishes bare incitement to crime more lightly than the deed itself in its perpetrator, and on the other hand the author and instigator more severely than the accomplice, as perpetrator of someone else's crime, and indeed the man who has been carried away unexpectedly by the great force of lust or anger more lightly than the one who is less troubled by them and has carried everything out with premeditation. The same view must be taken of a difference in outcome. The man who despoils a wayfarer of half a drachma deserves the same punishment, if he had his eye on a large haul, as the one who has snatched a talent; and on the other hand the man who has taken a moderate amount when he could take a whole mass of gold deserves a smaller punishment than the one who has snatched a purse with a handful of asses in it.[222] Indeed Bodin relates these geometric distinctions to the philosophical evaluation of charges; but punishments for which examples are established publicly, and which have a bearing on public order, he exempts from that all-embracing equality, so that deeds which happen before the citizens' eyes attract public attention more frequently, but inclinations which miss their aim are passed over unpunished, not being widely noticed, or punished more lightly than was geometrically fair, so that less cruelty is attributed to the judge. So much respect for the public good has a certain flavor of the harmonic.

XX. In giving judgment and in debates the ratio of the votes is arithmetical, if they are to be counted, and geometrical if they are

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[222] In classical Athens a talent was about 58 lbs of silver and so worth a considerable sum of money. A mina was a sixtieth of a talent and a drachma a hundredth of a mina. The name drachma was also given to the coin of the same weight of silver, which was therefore of significant but not very great value—the equivalent of, say, 50p in modern British currency. The *as* was a Roman coin of low value, which might be thought of as equivalent to a modern penny.

to be weighed, either by the importance of the one who casts them, or by the value of the arguments. There was a mixture which was harmonic (in Bodin's sense) among the Romans, for their consulars[223] explained their opinion in their speeches, and the rest voted with their feet for the opinion of which they approved. To me what came about by the law of necessity (for there would not have been enough time to hear everybody) does not seem comparable with harmonic proportion.

The law of votes.

XXI. 1. For justice in exchanges Aristotle sought arithmetic equality.[224] Not even here does Bodin say that a geometrical relationship should be barred, though his examples seem confused. 1. For instance : someone who has given offence publicly has also injured a private person, he shall in fact give satisfaction to the private person separately on a basis of arithmetic equality, and no greater restitution shall be made in relation to his means by a nobleman than by a peasant. However, the former will atone on a very severe scale in one way, the latter in another, preserving the geometrical proportion of their status. On this topic Aristotle agrees with Bodin.

Justice in exchanges.

2. However, for the goods of someone who is insolvent to be distributed according to the size of each debt without respect of creditors, provided that the debts were equal in time and nature, is I think merely a case of geometric proportion. As Aristotle did not deny that, and it seems to have been omitted by him through thoughtlessness, was correctly supplied by Bodin. However, it would be based on an arithmetic equality of repayment and debt, if the goods were sufficient. However, the superior magistrate is not completely forbidden to use harmonic tempering, and prefer needy creditors, up to a point and according to the particular circumstances, to those who are better off.

Creditors.

3. It also has some harmonic flavor (in Bodin's sense) that by the custom of certain nations compensation for stolen goods is not paid straightforwardly but fourfold, and for some goods fivefold – equally indeed by every person, but unequally for different kinds of goods, in which the differing magnitude of the offence is reckoned by geometrical ratio, not the bare price of the goods.

Reparation for a theft.

4. The harmonic ratio is more evident in the compensation for losses which have not been inflicted maliciously, when much depends on the circumstances of the actions of each of the contracting parties, which sometimes as directed either by the laws or by the magistrate require equal and arithmetic compensation, sometimes none, and on occasion compensation which is moderated by what is just and good. The situation also seems to be of this kind when the superior mag-

Losses.

[223] In classical Rome the real authority lay not so much with the two consuls, who were the chief magistrates but still comparatively young men, as with the senior consuls who dominated the senate.

[224] See note 211 above.

istrate does not allow all to resort to the law on an equal footing between themselves, nor in all cases, but according to what the circumstances were, he either gives them arbitrators himself, or permits the choice of them to the litigants, so that consideration may be given either to the interests of both parties or to those of the state.

Arbitrators.

5. Even the charging of interest is viewed from the aspect of the justice of exchange; but it is not permitted equally to every kind of men by all legislators. However, by a geometric mixture, unequal charges are permitted to persons of different orders, equal charges to individual members of the orders. This combination is for Bodin harmonic.

The charging of interest.

6. I shall copy an extraordinary exception to arithmetic equality of exchange in the words of Bodin. "A physician often takes 500 gold pieces for removing a stone from a rich man, but from a needy man 10 or 5"; (or nothing, according to the formula of the Hippocratic oath)[225] "but if they were to follow the arithmetic or geometric relationship in every case, the one in fact would die of the stone, the other of hunger: but in harmonic ratio in fact the wealth of the latter and the health of the former" (not I think with the same certainty) "are compared."

Remuneration.

7. In this way in fact even judges have customarily assessed their effort and their fee for deciding cases. For often the difficulty of the least important cases is greater than if the question concerns very weighty matters, but is unprofitable for the judges. It is therefore just that from the judgment of claims for very considerable amounts a little more should be demanded as a fee, that is to say in the situation where the judges do not get a sufficient salary from the republic, but with the intention of reducing the number of lawsuits they are instructed to seek recompense for the pains they have bestowed from the contributions of the litigant parties. However I leave this harmonic part-song to its author Bodin as a Frenchman: among us Germans justice in the chief states and provinces is kept far away from meanness of that kind, and it is not lawful to demand anything beyond what is prescribed by law.

The salary of a judge.

8. The equitable distribution of inheritances among persons of each sex is carried out by arithmetic laws, so that those who are equal in origin are taken as equal in their proportions of the estate left by the same father. But when the interests of the state are considered, estates fall to the menfolk, as they are fitted for the necessary military service, the money to the women and to persons who are consecrated to God; and when the first born are preferred to the rest by either

The division of inheritances.

[225] The Hippocratic Oath itself does not mention the fees which a physician ought to accept for giving treatment but merely that he should not take any fee for teaching medicine to the offspring of his teachers. The recommendation not to charge very much when a patient is unable to pay is in another work of Hippocrates, *On the Physician* (*Corpus medicorum graecorum*, ed. J.L. Heiberg [1927], I, 32, 4ff; see L. Edelstein, *Ancient Medicine* [Baltimore, 1967], p. 99).

the larger portion or possession of the whole inheritance, provision is made for the rest either by portions which are equal to each other or by a bequest or simply by a life interest so that an image of leadership, which is useful for the state, is set up in families. That comes about by harmonic ratios, which are sometimes very harsh and discordant, taken individually, inasmuch as arithmetic equality is disturbed; but they work together for the preservation of the state as a whole.

XXII. Under popular rule, which is shaped in accordance with arithmetic equality, and under aristocratic rule, with geometric similarity, nevertheless some account is often taken of tempering. For instance when the people itself is in control, of its own accord it grants honors, magistracies, and prelacies chiefly to patricians; or the aristocrats give the plebs a share in some of the honors, concede very rewarding public duties to the plebeians alone, penalize very severely wrongs inflicted on them by aristocrats, bestow on them freedom to enjoy their pleasures, which in itself has a flavor of the popular state, and permit a certain right of voting to the plebs, in naming candidates from the class of aristocrats to a certain number, from which the aristocrats themselves afterwards entrust magistracies to those they wish. Forms of state.

As long as it is by the will of the order which holds the supreme rule this collaboration stands, and a delightful harmony of civil concord endures; but if it is revoked, first complaints emerge from the part which has been deprived, then discords, as if the harmony has been disturbed, and finally either rule is transferred or the whole state succumbs to its enemies. The same point must be maintained about the royal situation, with which harmonic ratio is closely involved. In it one man alone possesses the supreme right to rule to the extent to which he surpasses all others; the rest carry on arithmetic equality under him.

Therefore in this political pattern can be the basis of the three kinds of government, which we touched on above in Section IX; but there we were dealing with the thing itself, here we are dealing with persons, or with the orders which are the limbs of the state. For if the king shares out all official duties equally, with no distinction between the nobility and the plebs, there will be a moderate government which is popular and similar to arithmetic, and unbecoming to supreme majesty, inasmuch as he himself who is supreme will be coupled with the lowest dregs, without the interposition of a middle order, though nature does love that. But citizens of noble quality will depart from the kingdom if there is no respect for birth.

But if he bestows everything on the nobility, shutting out the plebs entirely, this will be a geometric pattern of government, but dangerous, and without the pleasure of harmony; for the people, which is mighty in numbers, will eventually be offended and will despoil the nobility of their magistracies and honors, the king himself of his rule.

And even though the regent may more cleverly differentiate the orders into a greater number of degrees, and also differentiate their duties, yet if he keeps for each order its own, as at Rome the tribunate belonged solely to the plebeians, the consulate only to the patricians, the orders will be alienated from each other and will not blend into one body with secure concord between them. For the duties of the inferiors will be despised along with the order itself; so much so that at Rome a patrician could not hold the tribunate unless he abjured his nobility, and when the plebeians had eventually extorted the consulate, yet the plebeian consuls were themselves few in number and thereafter despised the plebs.[226]

Bodin puts forward the idea of these arguments in numbers[227]; and chiefly in 4, 6, and 7. here 4 represents the king, 6 the nobility, and he mentions that there is harmony between them; but 7 represents the plebs, and he rightly states that this number is discordant with both the earlier ones. He could better have used 4, 6, and 9. For as 6 makes a harmony with 9, and 4 makes no harmony with the same 9, so for the nobility it is easier to associate with the plebs than with the king; and as 6 is the geometric mean between 4 and 9, and concordant with both, so the nobility is interposed as a bond between the king and the plebs who are the lowest.

On the other hand he represents the aristocratic pattern of governing the kingdom with numbers which are in discontinuous geometric proportion, 3, 6, 5, and 10. The discontinuity he compares to dissension in the state, caused by the orders being too rigorously shut apart; and not inappropriately. But he does wrong to hold this discontinuity guilty of discord. For all these numbers, 3, 6, 5, and 10, make a common harmony. For the geometrical relationship does not in any way relate to the production of harmony, as has often been said: discontinuity is in no way a distinguishing feature of that relationship. And although as if correcting himself he does submit on the same page another example of numbers in discontinuous proportion, intended for the same purpose, that is to say 2, 4, 9, and 18, in which there is in truth dissension between the sounds symbolized by 2 and 4 and those symbolized by 9 and 18, yet this musical discord does not depend on the geometrical discontinuity, but on its own proper causes.

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[226] See note 209 above. During the struggle of the plebeians against the patricians in the early Roman Republic, the plebeians won for themselves the right to elect magistrates called tribunes, eventually ten in number, whose persons were inviolable and who had the right to veto any action of any other magistrate. When the plebeians won the right to hold the consulate, and the original function of the tribunes was no longer required, the office nevertheless continued, though naturally it could be held only by plebeians. Occasionally a patrician would arrange for himself to be adopted by a plebeian so that he could hold the tribunate, as did P. Clodius Pulcher in the time of Cicero, in 52 BC. Kepler also uses such terms as patrician and plebeian to refer to the corresponding classes of his own time.

[227] Bodin, *De republica* (1591), 1211–1213.

that is to say from the fact that the side of the nine-sided figure is not among those which are (knowable). Finally he expresses the tempering of both patterns of government in the royal state of affairs, that is to say in such a way that with the duties of the nobility some plebeians are involved, but few, and those with resources adequate for the task, or who have the recommendation of excellence or something else; and that sometimes one of the less well off nobles is put in charge of a profitable plebeian function, and casts a lustre over it by the splendor of his birth, so that in the future it is more acceptable to the plebs: so that a commission of two may consist of a noble and a plebeian, as a consolation to both orders; that courts may consist of persons from both orders; that in debates about the public good the needy may be mingled with the wealthy; that those who cultivate merely excellence and piety may not get everything, but some things may also be reserved for the brave, some for the gifted, some for the wise, some for those mighty in experience, and for each those things for which he supplies fitting endowments and the most resources; this tempering, I say, Bodin expresses in the numbers 4, 6, 8, and 12, thinking, as he himself declares, that he is putting forward a model of a continuing relationship, but as a continuous geometrical proportion. Again he is mistaken; for 4 is not to 6 as 6 is to 8, but the truth is that there is a shared concord between all four of the numbers 4, 6, 8, and 12, or in the simplest terms 2, 3, 4, and 6, though admittedly there is a different concord between 2 and 3, also between 3 and 4, and lastly between 2 and 4 or 3 and 6. Therefore whatever similarity the numbers have to the state they have to the extent that independently of music they are associated with geometric or arithmetic progressions, or with a mixture of both. Or on the other hand if the harmonic proportions of numbers bring any light to bear on the understanding of politics, they do it on their own account, independently of any relation with geometric proportions.

The tempering of royal government.

Certainly if I had acquired knowledge of the state, and was dealing with politics in this book, making a distinction between the subjects I should interpolate the whole of the passage from Bodin in such a way that not only would Aristotle's views have stood intact but Bodin would have learnt from this *Harmony* of mine, independently of Aristotle, how to be a better political philosopher. I should say that the condition of the state and the pattern of its government were one thing, and the administration of justice another, for they differ as part and whole; just as in mathematics geometric and arithmetic proportions in numbers are one thing, and musical harmonies expressed in numbers another. Certainly those who control things establish judgments, whether they be one, or few, or all; but the same people are occupied with safeguarding the state, and rarely administer justice themselves, except when it is legitimate for them to use their discretion unfettered by the laws. On the other hand superior offices are rarely entrusted to judges; rather laws are prescribed by which they are bound. Thus

True harmony is the Idea of the best possible state.

I should relegate geometric and arithmetic progressions, along with Aristotle, to the administration of justice, without any admixture of consideration for the harmony between parts which happens to occur in them. For the temperings which up to this point Bodin wished for in the administration of justice are eventually devolved on the supreme regent of the state himself in appeals. And in that case when he departs from the proportions which Aristotle gave to judgments and to justice, he is no longer judging but is exercising a higher office, safeguarding the state and its individual limbs. Certainly to this regent, whether he be king, or the aristocracy, or the entire people, I should recommend harmonic proportions, with no reference to a geometric relationship, no reference to arithmetic equality, but reference simply to the harmonies; and I should command those subtle relationships in harmonic parts, and similarly that inflexible administration of both kinds of justice, to yield place to the supreme rule and safety of the state. For God's sake, what a crop of arguments would there be, if anyone wanted to proceed to compare them point by point! For consider the nature of each thing. The geometric relationship is common to numbers and to continuous quantity, in such a way that it comes secondarily to numbers, in cases, of course, where their configuration allows the quantities to be expressed in number. It then very often happens that the terms embracing the geometric relationship are inexpressible, and the further we go the more they degenerate from expressibility. On the other hand every harmonic proportion can be expressed in numbers. What else does this mean when applied to politics, but that the decisions of judges which are the most just and according to the yardstick of the laws and of justice should be turned inside out and subjected to the most detailed examination by those learned in the law; whereas the direction of the state, unfettered by such great compulsions, should be adapted to the general well-being at the will of the ruler, according to the circumstances, without a great commotion. Here the enquiry should have been divided into parts, as it deals with how the rough and ready basis of arithmetic progression, without any nobility in its pattern, depending on the dimensions of the material alone, should be related to plebeian dealings, employments, and ways of life, geometric similarity (observing one single law, and not caring at all whether the line on which it is modeled is the side of a pentagon or a triangle, expressible or inexpressible) reflects simply the office of judge; but harmonic ratios, arising from many knowable regular figures, dovetailing into one those which are different in kind, represent the many distinct duties of the ruler, each one of which has its own special law, segregated from the rest. Here in agreement with Ptolemy (below in the appendix to Book V), of the kinds of harmonies and melody, the soft would in fact be given to peace, the hard to war; and the dissensions between the kinds are the same as those in the state of both times. However, I leave these matters to others to take care of, whose style of life is more appropriate for

Geometric and arithmetic proportions do not relate to the basis of the state of affairs, as do harmonic ones, but only to justice.

Peace from the soft kind, war from the hard kind.

them: I myself now hurry to the themes belonging to my profession which follow. Thus I shall first deal with Bodin. For I am arriving at the end of his book and at the same time at the basic starting point of this third book of mine.

XXIII. For now Bodin[228] is passing completely to music, as it has been described by me separately from geometric proportions, and, as I also decide to do, fitting it to politics; and he gives no more attractive argument in his whole book than about the pattern of the kingdom expressed in the numbers 1, 2, 3, and 4. As they are the first by nature, and unity indeed is the basic starting point of the numbers, so it is expressed in the point which is the basis of quantities, and thence their three kinds, 2 for a line, 3 for a surface, 4 for a body, almost in the same way as at the beginning of this Book III I reviewed from the Pythagorean theory, have been compared with them by a happy stroke of genius. Secondly, for him unity is the king, the representative in the state of God the omnipotent. 2 is the sacred order; 3 the military or knightly; 4 the people, and it is in fact in all respects fourfold: peasants, scholars, merchants, and workers. He reviews the harmonies comprised among these numbers in the same way as I have used above; but he has a pointless reason for being afraid of going on to the fivefold, as if the harmony would be disturbed, which as is evident from the whole of this book is false. There would certainly be other valid reasons for stopping at the fourfold. Furthermore he shows from Plato that this pattern of the state is revealed in man, as in it are 1. the mind; 2. the faculty of reason, which communicates itself in syllogisms; 3. the angry faculty; and 4. the libidinous faculty. Plato also shows that their seats in the brain, in the breast and in the belly are also separated by harmonic intervals,[229] and that the ruling power is related to the first of them, and has possession of most of it, the Council to the second, the military order to the third, and the plebs to the fourth. Hence the four pillars of states, Justice, Wisdom, Courage, and Temperance, truly called the cardinal virtues are entrusted particularly each to its own order. And so much of a musician did

The idea of the kingdom in numbers.

The soul.

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[228] Bodin, *De republica* (1591), 1215–1221.

[229] This is a slightly garbled version of Plato, *Timaeus*, 68–71. Plato there suggests that while the immortal soul of man was placed above the neck, there was also created another, mortal soul, placed below the neck. The part of the mortal soul which contained courage, passion, and quarrelsomeness was placed above the midriff in the chest, and the part which was concerned with bodily appetites, such as desire for food and drink, was placed below the midriff in the belly. However, Plato does not mention harmonic intervals in this context. Kepler may also be thinking of the threefold division of the soul which Plato generally makes, distinguishing between the reasoning part, the spirited part and the desires. In the *Republic* these three parts are shown to correspond with the philosopher-rulers, the soldiers and the workers respectively; and justice is found when each part of the soul or each class in the state performs its own function harmoniously. Thus there is nothing in Plato that quite corresponds with Bodin's fourfold division.

Bodin become, dismissing geometry and arithmetic, that he did not even forget dissonances,[230] for reflecting them he recommends that sometimes magistracies, honors, and other rewards of excellence be allowed to someone who does not deserve them, with no unfair intention, because from the opposition of a single dissonance the harmony of the rest is recognized all the better, and devoured all the more eagerly. Thus all those who guard against and denounce the glossing over of this kind of benefits to the state should become more and more accustomed to shunning similar faults and to the pursuit of virtue.

The world.

XXIV. Eventually Bodin compares the kingdom which he has described with the actual world, showing how God the creator has embellished this work of his by joining the ratios of equal and of similar in one concerted harmony. I agree with his purpose, as much as anyone; and what he or the preceding philosophers have not even touched on, which concerns the most accurate harmonic tempering of certain motions, I supply in the books which follow, and bring to light by the clearest demonstrations. However, I do not cease, even at the endpoint, to contradict Bodin when his mind is wandering; for no true harmonies are formed by the mixture of the equal and similar ratios, as has often been said. Therefore Bodin, thinking that he is demonstrating harmonies in the world, is rather demonstrating geometric means, and equalities, and similarities: 1. Harmonic ones indeed in the faculties of the soul; but those are only symbolic, and not exhibited by any clear demonstration in some solid representation, such as mathematics loves, inasmuch as it cannot be understood how there are in the soul quantities, as something very closely and immediately subject to harmonies. 2. Equality between the body of the world, and the universal force of matter. 3. Similarity (qualitative from the figures rather than relative to the size of the terms) between the eternal archetype and the shape of the world. In this passage Bodin touches my heart by referring to the theme of my *Secret of the Universe*, though in ignorance.

In every faculty of it of course the actual number which Bodin assigns to it.

Yet how much more plausibly is it demonstrated in Book IV of my *Epitome of Copernican Astronomy* that there is this mixture of the equal and the similar from the chief parts of the world, from the Sun which is the mover, the sphere which marks the position of the fixed stars, and the intermediate part, which is allotted to the moving bodies. For equality has been established in the parts of matter, so that there is just as much of it in the Sun as in the circle of the outermost sphere, or in the intermediate space. But arithmetic equality is due to matter, as to the people. On the other hand there is geometrical similarity

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[230] As is clear from his discussion of polyphonic music in Chapter 16, Kepler recognized the role of dissonances in harmony, provided they were drawn from melodic intervals, in order to add color. However, his theoretical principles naturally led him to assume that all enjoyment of music must come from consonances, and in dealing with the origin of the consonances he speaks as if this were the case.

of proportion in the density of the three bodies; and again there is another between the diameters of the Sun which is the mover, the region of the moving bodies, and the furthest unmoving sphere which marks their place. 4. Furthermore geometric means in a sense, such as that between the two kinds of motion, one eastwards, one westwards, and a third motion of trepidation, but of these the latter two are established according to Copernicus' opinions only in the eye of the imagination. 5. Between earth and tufa, clay. 6. Between metals and stones, coppery soil. 7. Between plants and stones, corals. 8. Between animals and plants, zoophytes. 9. Between quadrupeds and fishes, amphibians. 10. Between birds and fishes, flying fishes. 11. Between men and brutes, monkeys, or women according to Plato.[231] 12. Between beasts and angels, man, mortal in body like beasts, immortal in mind like angels. 13. Between the heavenly dwelling places of the blessed and the region of the elements, the starry heaven. 14. Add to these from the beginning of this book a pair of means, that is to say air and water between fire and earth; and so on.

But a considerably better reminder comes from those things which in the government of the world have the ratio of discords: faults in the soul; monsters among animals; eclipses in the heaven; inexpressibles in geometry (which, however, as they arise from the necessity of quantitative matter, in the following books will be far more correctly compared with the variety of the celestial motions); in the works of providence examples of divine anger and vengeance; among rational beings the devil. All of which are arranged by God the supreme Regent for a good end, and the most complete harmony of all things. To Him every creature that can breathe brings the most fitting sacrifices of praise with unceasing exercise of piety; and I myself indeed, if it should seem good to Him that I should not die but shall live, shall in the following books declare the works of the Lord.

End of Book III.

[231] The reference appears to be to Plato, *Timaeus*, 90E–91D.